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Optimization of energy functional in abstract spline interpolation: criteria and some results for splines with tension

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The aim of this paper is to introduce some formal procedures for the optimization of parameters in energy functional for variational spline interpolation. For the abstract splines with the tension we obtain the representation formula which shows the structure of dependence of spline with the tension with respect to tension parameters. It permits us to understand the structure of aim functions we need to minimize.

1. General approach for the optimization of energy functional in variational spline approximation

Let X, Y, Z_1, Z_2 be real Hilbert spaces, $A_1 : X \to Z_1$, $A_2 : X \to Z_2$ be linear bounded operators, and $z_1 \in Z_1$, $z_2 \in Z_2$ be the fixed elements.

Let us consider some set $\mathfrak{M} \subset R^n$ and the parametric family of linear bounded operators $T_\alpha : X \to Y$, $\alpha \in \mathfrak{M}$. We suppose that the following spline interpolation problems

$$\sigma_\alpha^{(1)} = \arg \min_{x \in A_1^{-1}(z_1)} \|T_\alpha x\|_Y^2, \tag{1}$$

$$\sigma_\alpha^{(2)} = \arg \min_{x \in A_2^{-1}(z_2)} \|T_\alpha x\|_Y^2, \tag{2}$$

$$\Sigma_\alpha = \arg \min_{x \in A_1^{-1}(z_1) \cap A_2^{-1}(z_2)} \|T_\alpha x\|_Y^2 \tag{3}$$

are uniquely solvable for every $\alpha \in \mathfrak{M}$. Here

$$A_1^{-1}(z_1) = \{x \in X : A_1 x = z_1\}, \quad A_2^{-1}(z_2) = \{x \in X : A_2 x = z_2\}. \tag{4}$$

Thus we suppose that [1]

$$A_1^{-1}(z_1) \neq \emptyset, \quad A_2^{-1}(z_2) \neq \emptyset, \quad A_1^{-1}(z_1) \cap A_2^{-1}(z_2) \neq \emptyset,$$

and for every $\alpha \in \mathfrak{M}$ the image of T_α is closed in X, $\ker T_\alpha \cap \ker A_i$ contains only zero vector, $\ker T_\alpha + \ker A_i$ is closed in X, $i = 1, 2$.

We formulate now few criteria of the optimal choice of the parameter α. In every criterion we need to minimize some reasonable aim function $\varphi(\alpha)$, $\alpha \in \mathfrak{M}$:

a) One-sided criteria: $\varphi(\alpha) = \|A_2\sigma_\alpha^{(1)} - z_2\|_{Z_2}$ or $\varphi(\alpha) = \|A_1\sigma_\alpha^{(2)} - z_1\|_{Z_1}$;

b) Two sided weighted criterion: $\varphi(\alpha) = c_1\|A_2\sigma_\alpha^{(1)} - z_2\|_{Z_2} + c_2\|A_1\sigma_\alpha^{(2)} - z_1\|_{Z_1}$, $c_1 > 0$, $c_2 > 0$, $c_1 + c_2 = 1$;

c) Cross criterion: $\varphi(\alpha) = \|\sigma_\alpha^{(1)} - \sigma_\alpha^{(2)}\|_X$;

d) Hybrid criteria: $\varphi(\alpha) = \|\Sigma_\alpha - \sigma_\alpha^{(1)}\|_X$ or $\varphi(\alpha) = \|\Sigma_\alpha - \sigma_\alpha^{(2)}\|_X$.

In practice, each of these criteria can be replaced by its residual version i.e., the parameter α is acceptable if the value of aim function becomes sufficiently small.

2. Abstract splines with the tension

General optimization problem for energy functional generated by arbitrary family of energy operators $\{T_\alpha\}_{\alpha \in \mathfrak{M}}$ may be extremely complicated both from theoretical and numerical points of view. Therefore, we consider here the simplest case of the family $\{T_\alpha\}$ linearly depending on parameters.

Let $\mathfrak{M}$ be the positive octant in R^n,

$$\mathfrak{M} = [0, +\infty) \times [0, +\infty) \times \ldots \times [0, +\infty), \tag{5}$$

and $T_0, T_1, \ldots, T_n$ be linear bounded operators from X to the Hilbert spaces $Y_0, Y_1, \ldots, Y_n$. Let $\alpha \in \mathfrak{M}$ be a fixed vector with the components α_1, α_2, $\ldots, \alpha_n$. We formulate the following spline interpolation problem: find $\sigma_\alpha \in X$ under interpolation constrain $A_1\sigma_\alpha = z_1$ which minimizes the energy functional

$$\|T_0\sigma_\alpha\|^2_{Y_0} + \sum_{k=1}^{n} \alpha_k \|T_k\sigma_\alpha\|^2_{Y_k} \overset{\text{df}}{=} \|T_\alpha\sigma_\alpha\|^2_Y. \tag{6}$$

Here $Y = Y_0 \times Y_1 \times \ldots \times Y_n$ is the new Hilbert space with the scalar product

$$(u, v)_Y = \sum_{k=0}^{n} (u_k, v_k)_{Y_k}$$

and with the corresponding Hilbert norm, and $T_\alpha : X \to Y$ acts by the rule

$$T_\alpha x = \Big[T_0, x, \sqrt{\alpha_1}\, T_1 x, \ldots, \sqrt{\alpha_n}\, T_n x\Big]. \tag{7}$$

Splines of this type we call "abstract splines with the tension". In accordance with the general theory, to find the spline σ_α we need to solve the operator system

$$T_\alpha^* \sigma_\alpha + A_1^* \lambda_\alpha = 0, \qquad A_1 \sigma_\alpha = z_1. \tag{8}$$

Here $\lambda_\alpha \in Z_1$ is the Lagrangian multiplier, and $T_\alpha^* T_\alpha : X \to X$ is of the form

$$T_\alpha^* T_\alpha = T_0^* T_0 + \sum_{k=1}^{n} \alpha_k T_k^* T_k. \tag{9}$$

3. How spline with the tension depends on parameters?

To analyze the structure of aim functions $\varphi(\alpha)$ in optimization criteria with respect to parameters of tensions $\alpha_1, \alpha_2, \ldots, \alpha_n$ we need to analyze how σ_α depends on these parameters.

It is easy to see that the spline with the tension σ_α can be always represented in the form

$$\sigma_\alpha = \sigma_0 - n_\alpha,$$

where interpolating spline σ_0 corresponds to zero coefficients $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, and n_α belongs to $\ker A_1$. If we consider some basis system $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ (may be infinite) in the linear space $\ker A_1$, the minimization of energy functional with respect to the coefficients b_i in the expansion $n_\alpha = \sum_j b_j \varphi_j$ leads to the system

$$\left(B_0 + \sum_{k=1}^{n} \alpha_k B_k\right) \vec{b} = \sum_{k=1}^{n} \alpha_k f^{(k)}, \tag{10}$$

where $B_0 = B_0^* > 0$, $B_k = B_k^* \geq 0$. Therefore, the spectral analysis of this system can be applied with the eigenvalues and vectors of the generalized eigenvalue problems

$$B_k u^{(k)} = \lambda^{(k)} B_0 u^{(k)}, \quad k = 1, 2, \ldots, n. \tag{11}$$

The following theorem is valid.

Theorem. *If generalized eigenvalue problems* (11) *have common basis of eigenvectors, then the following representation formula for the spline* σ_α *with the tensions takes place*

$$\sigma_\alpha = \sigma_0 - \sum_j \frac{\sum_{k=1}^{n} \alpha_k \lambda_j^{(k)}}{1 + \sum_{k=1}^{n} \alpha_k \lambda_j^{(k)}} \cdot C_j(\sigma_0) \psi_j, \tag{12}$$

where $C_j(\sigma_0)$ is independent of α linear functionals and ψ_j are normalized in scalar product $(T_0^*T_0u, v)_X$ basis elements in $\ker A_1$. *These elements are also independent of α. Moreover, $\lambda_j^{(k)} \geq 0$, $k = 1, 2, \ldots, n$, $j = 1, 2, \ldots$.*

Remark. If $n = 1$, then the basis of eigenvector is always "common" and one-parametric representation does always exist:

$$\sigma_\alpha = \sigma_0 - \sum_j \frac{\alpha\lambda_j}{1 + \alpha\lambda_j} C_j(\sigma_0)\psi_j.$$

Thus, this theorem allows us to clarify the structure of various aim functions in various optimization criteria and to choose numerical methods for the optimization. In practical computations, the optimization of parameters in energy functional shows us high efficiency and essential improvement of the results. But the numerical methods for this optimization requires further theoretical researches.

References

[1] Bezhaev A.Yu., Vasilenko V.A. Variational Spline Theory. – Novosibirsk: NCC Publisher, 1993. – (NCC Bulletin, Series Num. Anal., Special issue 3).

[2] Vasilenko V.A. Optimization of energy functional for variational splines // NCC Bulletin, Series Num. Anal. – Novosibirsk: NCC Publisher, 1996. – Issue 7. – P. 101–105,

[3] Vasilenko V.A., Elyseev A.V. Abstract splines with the tension as the functions of parameters in energy operator // Sib. J. Num. Math. – 1998. – Vol. 1, № 4. – P. 301–311 (in Russian).