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On one system of the Burgers equations arising in the two-velocity hydrodynamics

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Abstract. A system of the Burgers equations of the two-velocity hydrodynamics is obtained. We consider the Cauchy problem in the case of a one-dimensional system. The estimate of the stability of the solution is obtained. We have obtained a formula for the Cauchy problem for the one-dimensional system of equations that arises in the two-velocity hydrodynamics. It is shown that with disappearance of the kinetic friction coefficient, which is responsible for the energy dissipation, this formula turns to the famous Cauchy problem for the one-dimensional Burgers equation. The existence and uniqueness of solutions to the Cauchy problem for the one-dimensional systems of the Burgers type are proved using the method of weak approximation.

Keywords: two-velocity hydrodynamics, Burgers type system, Florin-Hopf-Cole transformation, method of weak approximation.

1. Introduction

In recent decades, mathematicians have become increasingly interested in the problems associated with the behavior of solutions to systems of partial differential equations with a small parameter in high derivatives with allowance for the kinetic parameters. These problems arose from the physical applications, mostly from contemporary hydrodynamics (compressible multiphase fluids with low viscosity). An analog to the Burgers equation arises, for example, in studying a weak nonlinear one-dimensional acoustic wave moving with a linear velocity of sound. In this case, nonlinear velocity terms in the system of the Burgers equations come from the sound velocity depending on the amplitude of the sound wave, on the second derivative terms and on the difference in the velocities representing the attenuation of sound waves associated with energy dissipation. In other words, these terms provide the continuity of solutions and are dissipative processes associated with the production of entropy. These terms in turn provide non-roll waves [1]. The system under consideration is a special case of a two-velocity system of hydrodynamics equations [2–6].

A one-dimensional analog of the Navier–Stokes equations for a compressible fluid can be considered as a system of the Burgers equations which is a system of nonlinear convection-diffusion equations [7]

$$u_t + uu_x = \nu u_{xx} - \tilde{b}\,(u - \tilde{u}), \tag{1}$$
$$\tilde{u}_t + \tilde{u}\tilde{u}_x = \tilde{\nu}\tilde{u}_{xx} + b\,(u - \tilde{u}), \tag{2}$$

where the quantities u and $\tilde{u}$ can be regarded as velocity subsystems with the dimension $[x]/[t]$, constituting two-velocity components with the corresponding partial continuum densities ρ and $\tilde{\rho}$, $\bar{\rho} = \tilde{\rho} + \rho$ is the common density of the continuum, $\tilde{b} = \frac{\tilde{\rho}}{\rho} b$, $b > 0$ is the coefficient of friction with the dimension $1/[t]$, which is the analog to the Darcy factor for porous media. The positive constants ν and $\tilde{\nu}$ play the role of kinematic viscosities subsystems of the dimension $[x]^2/[t]$.

A two-velocity system of hydrodynamic equations and a system of the Burgers equations have much in common. For example, the quadratic nonlinearity with respect to the terms u and $\tilde{u}$ with advective terms corresponding to the sound, depending on the amplitude of the sound waves and linear viscosities ν, $\tilde{\nu}$, the coefficient of friction b of the right-hand side is responsible for the attenuation of the sound waves [1]. With regard to the properties of the solutions, they are totally different. The system of the Burgers equations in the vanishing coefficients ν, $\tilde{\nu}$, b is formed of both strong (shock waves), and weak discontinuities, while the solution of the system of two-velocity hydrodynamics does not have such features. However, the range of applicability of this system is by no means limited to the above example. These systems arise in many problems, thus confirming their significance.

2. The Cauchy problem for the system of the Burgers type equations

For system (1), (2) in the domain $\Omega_{[0,T]} = \{(t, x) :\ 0 \le t \le T,\ x \in R\}$ let us consider the Cauchy problem with the following initial data

$$u|_{t=0} = u_0(x), \quad \tilde{u}|_{t=0} = \tilde{u}_0(x), \quad x \in R. \tag{3}$$

We are interested in the classical solution of the Cauchy problem for the system of the Burgers equations (1), (2), namely, u, $\tilde{u} \in C^{1,2}(\Omega)$ is the class of functions once continuously differentiable with respect to t and twice continuously differentiable with respect to x.

Theorem 1. *Let u_0, $\tilde{u}_0 \in C^2(R) \cap W_2^1(R)$, where $W_2^1(R)$ is the Sobolev space. Then the Cauchy problem (1)–(3) has in the class $C^{1,2}(\Omega_{[0,T]})$ a unique solution, and the following estimate of stability holds*

$$\int_{\Omega_{[0,T]}} (u^2(t,x) + \tilde{u}^2(t,x))\,dx\,dt \le T\,\frac{\max\{b,\tilde{b}\}}{\min\{b,\tilde{b}\}} \int_{-\infty}^{\infty} (u_0^2(x) + \tilde{u}_0^2(x))\,dx. \tag{4}$$

Proof. Multiply both sides of equations (1) and (2) by u and $\tilde{u}$, respectively. After integration with respect to x and after simple transformations we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} u^2\,dx = -\nu\int_{-\infty}^{\infty}(u_x)^2\,dx - \tilde{b}\int_{-\infty}^{\infty}(u^2 - u\tilde{u})\,dx, \tag{5}$$

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \tilde{u}^2\,dx = -\tilde{\nu}\int_{-\infty}^{\infty}(\tilde{u}_x)^2\,dx + b\int_{-\infty}^{\infty}(u\tilde{u} - \tilde{u}^2)\,dx. \tag{6}$$

Hence, multiplying (5) by b, (6) by $\tilde{b}$ and summing the results obtained, after simple transformations we arrive at

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}(bu^2 + \tilde{b}\tilde{u}^2)\,dx \le 0.$$

Now, we can find the function $m(t) = \int_{-\infty}^{\infty}(u^2(t,x) + \tilde{u}^2(t,x))\,dx$ satisfies the inequality

$$m(t) \le \frac{\max\{b,\tilde{b}\}}{\min\{b,\tilde{b}\}} m(0).$$

Hence, integrating from 0 to T, we obtain (4). The uniqueness is proved in a standard way. □

3. The formula for solving the Cauchy problem for the Burgers type system of equations

Next, assume as in [8] that the Cauchy data $u_0(x)$, $\tilde{u}_0(x)$ for large $|x|$ satisfy the following conditions

$$\int_0^x u_0(\xi)\,d\xi = o(x^2), \qquad \int_0^x \tilde{u}_0(\xi)\,d\xi = o(x^2). \tag{7}$$

For simplicity, assume $\operatorname{supp} u_0, \operatorname{supp}\tilde{u}_0 \subset [0,\infty)$.

It is convenient to use the Florin–Hopf–Cole transformation

$$\phi(t,x) = \exp\left[-\frac{1}{2\nu}\int_{-\infty}^{x} u(t,\xi)\,d\xi\right], \quad \psi(t,x) = \exp\left[-\frac{1}{2\tilde{\nu}}\int_{-\infty}^{x} \tilde{u}(t,\xi)\,d\xi\right],$$

herewith the functions u and v are expressed in terms of the functions ϕ and ψ by the formulas

$$u = -2\nu\frac{\phi_x}{\phi}, \qquad \tilde{u} = -2\tilde{\nu}\frac{\psi_x}{\psi}.$$

In terms of the functions ϕ and ψ the system of dynamic equations (1) and (2) takes the form

$$\Big(\frac{\phi_t}{\phi}\Big)_x = \Big(\nu\frac{\phi_{xx}}{\phi}\Big)_x - \frac{\tilde{b}}{\nu}\Big(\ln\frac{\phi^\nu}{\psi^{\tilde{\nu}}}\Big)_x,$$
$$\Big(\frac{\psi_t}{\psi}\Big)_x = \Big(\tilde{\nu}\frac{\psi_{xx}}{\phi}\Big)_x + \frac{b}{\tilde{\nu}}\Big(\ln\frac{\phi^\nu}{\psi^{\tilde{\nu}}}\Big)_x.$$

Hence, after the integration with respect to x, we obtain

$$\phi_t = \nu\phi_{xx} - \frac{\tilde{b}}{\nu}(\nu\ln\phi - \tilde{\nu}\ln\psi)\phi + C_1(t)\phi, \tag{8}$$
$$\psi_t = \tilde{\nu}\psi_{xx} + \frac{b}{\nu}(\nu\ln\phi - \tilde{\nu}\ln\psi)\psi + C_2(t)\psi, \tag{9}$$

where $C_1(t)$ and $C_2(t)$ are arbitrary functions. Taking into account the behavior of ψ and ϕ at $x \to -\infty$, we conclude $C_1(t) = C_2(t) = 0$.

The Cauchy problem for system (8), (9) with the data

$$\phi|_{t=0} = \phi_0(x), \quad \psi|_{t=0} = \psi_0(x)$$

has the form

$$\phi(t,x) = \int_{-\infty}^{\infty} G_\nu(x,\xi,t)\phi_0(\xi)\,d\xi - \frac{\tilde{b}}{\nu}\int_0^t\int_{-\infty}^{\infty} G_\nu(x,\xi,t-\tau)[\nu\ln\phi(\tau,\xi) - \tilde{\nu}\ln\psi(\tau,\xi)]\phi(\tau,\xi)\,d\xi\,d\tau, \tag{10}$$

$$\psi(t,x) = \int_{-\infty}^{\infty} G_{\tilde{\nu}}(x,\xi,t)\psi_0(\xi)\,d\xi + \frac{b}{\nu}\int_0^t\int_{-\infty}^{\infty} G_{\tilde{\nu}}(x,\xi,t-\tau)[\nu\ln\phi(\tau,\xi) - \tilde{\nu}\ln\psi(\tau,\xi)]\psi(\tau,\xi)\,d\xi\,d\tau, \tag{11}$$

where

$$G_a(x,\xi,t) = \frac{1}{\sqrt{4\pi a t}}\exp\Big(-\frac{(x-\xi)^2}{4at}\Big)$$

is the fundamental solution of the one-dimensional heat equation.

Let us introduce the following functions

$$F(u,x,y,t) = \frac{(x-y)^2}{2t} + \int_0^y u(t,\eta)\,d\eta, \tag{12}$$
$$F_1(u,x,y,t,\tau) = \frac{(x-y)^2}{2(t-\tau)} + \int_0^y u(\tau,\eta)\,d\eta. \tag{13}$$

Note that the following equalities for any x, y, t, τ are valid

$$F(u_0, x, y, t) - F(\tilde{u}_0, x, y, t) = \int_0^y [u_0(\eta) - \tilde{u}_0(\eta)]\, d\eta,$$
$$F_1(u, x, y, t, \tau) - F_1(\tilde{u}, x, y, t, \tau) = \int_0^y [u(\tau, \eta) - \tilde{u}(\tau, \eta)]\, d\eta.$$

Differentiate both sides of equalities (9) and (10) with respect to x. After simple transformations we obtain

$$\begin{aligned}
\phi_x(t, x) = & -\frac{1}{2\nu} \int_{-\infty}^{\infty} u_0(\xi) G_\nu(x, \xi, t) \exp\left[-\frac{1}{2\nu}\int_0^\xi u_0(\eta)\, d\eta\right] d\xi + \\
& \frac{\tilde{b}}{2\nu} \int_0^t \int_{-\infty}^{\infty} \frac{x - \xi}{t - \tau} G_\nu(x, \xi, t - \tau) \exp\left[-\frac{1}{2\nu}\int_0^\xi u(\tau, \eta)\, d\eta\right] \times \\
& \int_0^\xi [u(\tau, \eta) - \tilde{u}(\tau, \eta)]\, d\eta\, d\xi\, d\tau, \\
\psi_x(t, x) = & -\frac{1}{2\tilde{\nu}} \int_{-\infty}^{\infty} \tilde{u}_0(\xi) G_{\tilde{\nu}}(x, \xi, t) \exp\left[-\frac{1}{2\tilde{\nu}}\int_0^\xi \tilde{u}_0(\eta)\, d\eta\right] d\xi - \\
& -\frac{b}{2\tilde{\nu}} \int_0^t \int_{-\infty}^{\infty} \frac{x - \xi}{t - \tau} G_{\tilde{\nu}}(x, \xi, t - \tau) \exp\left[-\frac{1}{2\tilde{\nu}}\int_0^\xi \tilde{u}(\tau, \eta)\, d\eta\right] \times \\
& \int_0^\xi [u(\tau, \eta) - \tilde{u}(\tau, \eta)]\, d\eta\, d\xi\, d\tau.
\end{aligned}$$

Hence, taking into account (12), (13) and the definition of the fundamental solution of the operator of conductivity we obtain

$$\begin{aligned}
\phi_x(t, x) = & -\frac{1}{2\nu\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} u_0(\xi) \exp\left[-\frac{1}{2\nu} F(u_0, x, \xi, t)\right] d\xi + \\
& \frac{\tilde{b}}{2\nu} \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\nu(t - \tau)}} \frac{x - \xi}{t - \tau} F_2(u, \tilde{u}, x, \xi, t - \tau, \tau)\, d\xi\, d\tau, \quad (14) \\
\psi_x(t, x) = & -\frac{1}{2\tilde{\nu}\sqrt{4\pi\tilde{\nu} t}} \int_{-\infty}^{\infty} \tilde{u}_0(\xi) \exp\left[-\frac{1}{2\tilde{\nu}} F(\tilde{u}_0, x, \xi, t)\right] d\xi - \\
& \frac{b}{2\tilde{\nu}} \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\tilde{\nu}(t - \tau)}} \frac{x - \xi}{t - \tau} \tilde{F}_2(u, \tilde{u}, x, \xi, t - \tau, \tau)\, d\xi\, d\tau. \quad (15)
\end{aligned}$$

In formulas (14) and (15) the following notation is used

$$F_2(u, v, x, \xi, t-\tau, \tau) = \exp\left[-\frac{1}{2\nu} F_1(u, x, \xi, t-\tau, \tau)\right] \int_0^\xi [u(\tau, \eta) - v(\tau, \eta)]\, d\eta,$$
$$\tilde{F}_2(u, v, x, \xi, t-\tau, \tau) = \exp\left[-\frac{1}{2\tilde{\nu}} F_1(v, x, \xi, t-\tau, \tau)\right] \int_0^\xi [u(\tau, \eta) - v(\tau, \eta)]\, d\eta.$$

Thus, we come to the following

Theorem 2. *Let $u_0(x)$, $\tilde{u}_0(x)$ be measurable functions that satisfy relation* (7)*. Then for the solution of the Cauchy problem* (1)–(3) *we have the formula*

$$
\begin{aligned}
u(t,x) = &\left\{\int_{-\infty}^{\infty} u_0(\xi)\exp\left[-\frac{1}{2\nu}F(u_0,x,\xi,t)\right]d\xi - \right.\\
&\left.\int_0^t\int_{-\infty}^{\infty}\frac{\tilde{b}\sqrt{t}}{\sqrt{t-\tau}}\frac{x-\xi}{t-\tau}F_2(u,\tilde{u},x,\xi,t-\tau,\tau)\,d\xi\,d\tau\right\}\Bigg/\\
&\left\{\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\nu}F(u_0,x,\xi,t)\right]d\xi - \right.\\
&\left.\int_0^t\int_{-\infty}^{\infty}\frac{\tilde{b}\sqrt{t}}{\sqrt{t-\tau}}F_2(u,\tilde{u},x,\xi,t-\tau,\tau)\,d\xi\,d\tau\right\},\\
\tilde{u}(t,x) = &\left\{\int_{-\infty}^{\infty} \tilde{u}_0(\xi)\exp\left[-\frac{1}{2\tilde{\nu}}F(\tilde{u}_0,x,\xi,t)\right]d\xi + \right.\\
&\left.\int_0^t\int_{-\infty}^{\infty}\frac{b\sqrt{t}}{\sqrt{t-\tau}}\frac{x-\xi}{t-\tau}\tilde{F}_2(u,\tilde{u},x,\xi,t-\tau,\tau)\,d\xi\,d\tau\right\}\Bigg/\\
&\left\{\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\tilde{\nu}}F(v_0,x,\xi,t)\right]d\xi + \right.\\
&\left.\int_0^t\int_{-\infty}^{\infty}\frac{b\sqrt{t}}{\sqrt{t-\tau}}\tilde{F}_2(u,\tilde{u},x,\xi,t-\tau,\tau)\,d\xi\,d\tau\right\}.
\end{aligned}
$$

Corollary. *Let $u_0(x)$, $\tilde{u}_0(x)$ satisfy the conditions of Theorem 2. Then for solving the Cauchy problem* (1)–(3) *we have the formulas*

$$
\begin{aligned}
u(t,x) = &\frac{\int_{-\infty}^{\infty} u_0(\xi)\exp\left[-\frac{1}{2\nu}F(u_0,x,\xi,t)\right]d\xi}{\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\nu}F(u_0,x,\xi,t)\right]d\xi} + \\
&\tilde{b}\frac{\int_0^t\int_{-\infty}^{\infty}\sqrt{1+\frac{\tau}{t-\tau}}\left(u(t,x)-\frac{x-\xi}{t-\tau}\right)F_2(u,\tilde{u},x,\xi,t-\tau,\tau)\,d\xi\,d\tau}{\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\nu}F(u_0,x,\xi,t)\right]d\xi},
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\tilde{u}(t,x) = &\frac{\int_{-\infty}^{\infty} \tilde{u}_0(\xi)\exp\left[-\frac{1}{2\tilde{\nu}}F(\tilde{u}_0,x,\xi,t)\right]d\xi}{\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\tilde{\nu}}F(\tilde{u}_0,x,\xi,t)\right]d\xi} - \\
&b\frac{\int_0^t\int_{-\infty}^{\infty}\sqrt{1+\frac{\tau}{t-\tau}}\left(\tilde{u}(t,x)-\frac{x-\xi}{t-\tau}\right)\tilde{F}_2(u,\tilde{u},x,\xi,t-\tau,\tau)\,d\xi\,d\tau}{\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\tilde{\nu}}F(\tilde{u}_0,x,\xi,t)\right]d\xi}.
\end{aligned}
\tag{17}
$$

Remark 1. With disappearance of the friction coefficient b (in the absence of dissipation energy due to friction) solution (16), (17) turns to the famous Cauchy problem for the Burgers equation [8].

4. The method of weak approximation for the Cauchy problem for the Burgers type system of equations

Let us consider problem (1)–(3) relative to the Cauchy data u_0, $\tilde{u}_0$, assuming that u_0, $\tilde{u}_0 \in C^2(R)$ and

$$\left|\frac{d^n u_0(x)}{dx^n}\right| \le c_n, \quad \left|\frac{d^n \tilde{u}_0(x)}{dx^n}\right| \le \tilde{c}_n, \quad x \in R, \quad n = 0, 1, 2,$$

where c_n, $\tilde{c}_n$ are some non-negative constants.

First, we consider the case of infinitely differentiable Cauchy data. Let us assume that u_0, $\tilde{u}_0 \in C^\infty(R)$ and

$$\left|\frac{d^n u_0(x)}{dx^n}\right| \le c_n, \quad \left|\frac{d^n \tilde{u}_0(x)}{dx^n}\right| \le \tilde{c}_n, \quad x \in R, \quad n = 0, 1, \dots \tag{18}$$

Following [9, 10], we use a weak approximation of the Cauchy problem (1)–(3) with the help of a series of problems

$$u_t^\tau = 3\nu u_{xx}^\tau, \quad \tilde{u}_t^\tau = 3\tilde{\nu}\tilde{u}_{xx}^\tau, \quad n\tau < t \le \left(n + \frac{1}{3}\right)\tau, \tag{19}$$

$$u_t^\tau + 3u^\tau u_x^\tau = 0, \quad \tilde{u}_t^\tau + 3\tilde{u}^\tau \tilde{u}_x^\tau = 0, \quad \left(n + \frac{1}{3}\right)\tau < t \le \left(n + \frac{2}{3}\right)\tau, \tag{20}$$

$$u_t^\tau = -3\tilde{b}(u^\tau - \tilde{u}^\tau), \quad \tilde{u}_t^\tau = 3b(u^\tau - \tilde{u}^\tau), \quad \left(n + \frac{2}{3}\right)\tau < t \le (n+1)\tau, \tag{21}$$

$$u^\tau(0, x) = u_0(x), \quad \tilde{u}^\tau(0, x) = \tilde{u}_0(x), \tag{22}$$

where $N\tau = t^*$, provided that N is integer exceeding 1, $n = 0, 1, \dots, N-1$, and the constant t^* satisfies equality (25) below.

Remark 2. When building the solution of (19)–(22), at the first fractional step, we solve the Cauchy problem for the heat equation, at the second fractional step, we solve the Cauchy problem for the transport equation

$$v_t + 3vv_x = 0, \tag{23}$$

and, at the third fractional step, we solve the Cauchy problem for systems of ordinary differential equations.

It is known that in the case of the Cauchy problem for equation (23) with the initial data

$$v(0,x) = v_0(x) \tag{24}$$

bounded together with their derivatives, the solution may be a gradient catastrophe, that is, there can exist $t_1 > 0$, such that the classical solution v of this problem would exist in the domain $\Omega_{[0,t_1)}$, be bounded in this domain, but the derivative v_x in the neighborhood of a point (t_1, x^0) is becoming unbounded: $v_x(t,x) \to \infty$ with $t \to t_1$, $x \to x^0$ [1, 11, 12].

It is easy to show that if $\left|\frac{dv_0(x)}{dx}\right| \le c_1$, the classical solution of problem (23), (24) in the domain $\Omega_{[0,t^*]}$ is bounded and the following estimate holds

$$|v_x(t,x)| \le \frac{c_1}{1 - 3c_1 t}, \quad (t,x) \in \Omega_{[0,t^*]},$$

where t satisfies the inequality $1 - 3c_1 t^* > 0$.

Let relations (18) be performed and the constants c_1, $\tilde{c}_1$ and t^* satisfy the conditions

$$1 - c_1 t^* > 0, \quad 1 - \tilde{c}_1 t^* > 0. \tag{25}$$

Then the solution u^τ, $\tilde{u}^\tau$ in the domain $\Omega_{[0,t^*]}$ exists and is bounded together with all its derivatives with respect to the variables t, x.

It is obvious that for any fixed τ, the solution of u^τ and $\tilde{u}^\tau$ of problem (19)–(22) is bounded regardless of the value τ:

$$|u^\tau(t,x)| \le c_0, \quad |\tilde{u}^\tau(t,x)| \le \tilde{c}_0. \tag{26}$$

Repeating the argument from [9], we can show the boundedness of the derivative solutions u^τ and $\tilde{u}^\tau$ of any order with respect to x:

$$\left|\frac{\partial^k u^\tau(t,x)}{\partial x^k}\right| \le C_k, \quad \left|\frac{\partial^k \tilde{u}^\tau(t,x)}{\partial x^k}\right| \le \tilde{C}_k, \quad (t,x) \in \Omega_{[0,t^*]}, \quad k = 0, 1, \ldots, \tag{27}$$

where C_k, $\tilde{C}_k$ are some positive constants such that $C_0 = c_0$, $\tilde{C}_0 = \tilde{c}_0$.

Independent of τ, from expressions (26), (27) and from equations (19)–(21) follow the estimates:

$$\left|\frac{\partial^{k+1} u^\tau(t,x)}{\partial t \partial x^k}\right| \le s_k, \quad \left|\frac{\partial^{k+1} \tilde{u}^\tau(t,x)}{\partial t \partial x^k}\right| \le \tilde{s}_k, \quad (t,x) \in \Omega_{[0,t^*]}, \quad k = 0, 1, \ldots$$

These estimates suggest that u^τ, $\tilde{u}^\tau$ and their derivatives with respect to x of any order are uniformly bounded and equicontinuous in $\Omega_{[0,t^*]}$. By the Arzela theorem, using the diagonal method we can choose subsequences $\{u^{\tau_k}\}$, $\{\tilde{u}^{\tau_k}\}$ of sequences $\{u^\tau\}$, $\{\tilde{u}^\tau\}$ converging in $\Omega_{[0,t^*]}$ to the functions u and $\tilde{u}$, respectively, together with all the derivatives with respect to x, uniform in every bounded region of the domain $\Omega_{[0,t^*]}$, whereby the functions u and $\tilde{u}$ have derivatives of any order with respect to x and satisfy the following relations

$$u(0,x) = u_0(x), \qquad \tilde{u}(0,x) = \tilde{u}_0(x),$$
$$\left|\frac{\partial^k u(t,x)}{\partial x^k}\right| \le C_k, \quad \left|\frac{\partial^k \tilde{u}(t,x)}{\partial x^k}\right| \le \tilde{C}_k, \quad (t,x) \in \Omega_{[0,t^*]}, \quad k = 0, 1, \ldots.$$

The uniqueness of the solution is proved in a standard way. Consequently, the sequences of the functions $\{u^\tau\}$, $\{\tilde{u}^\tau\}$ with $\tau \to 0$ uniformly converge in the domain $\Omega_{[0,t^*]}$ to u and $\tilde{u}$, respectively, together with all their derivatives. The case when u_0, $\tilde{u}_0 \in C^2(R)$ is proved using the average functions [13].

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