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The HLL method for solving the Riemann problem in compressible two-velocity media*

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Abstract. The paper considers a system of equations of motion of a two-phase medium with phase equilibrium in terms of pressure and temperature. The spectral analysis of the system was carried out for a one-dimensional case. Approximate numerical solutions of the Riemann problem about the decay of an arbitrary discontinuity were constructed using the HLL method, using the calculated characteristic values. A comparison with a single-phase analog of the system is made.

Keywords: two-phase fluid, two-velocity hydrodynamics, ideal fluid, HLL method, Riemann problem, characteristics.

1. Introduction

Mathematical models of heterophase multicomponent media are widely used to describe non-stationary processes in technological and natural systems. Models of two-velocity media allow investigating the mechanics of porous materials, granular media, mixtures of immiscible liquids, bubbly and boiling liquids, hydrothermal flows, unconsolidated mixtures, fluid-magmatic flows containing xenoliths. To implement such tasks, various methods for solving systems of nonlinear differential equations have been proposed: the Hirota bilinear method [1], the inverse scattering method [2], the Backlund transformation [3] or the Darboux transformation [4], the truncated Painleve expansion [5], the group analysis method [6], the balance method [7], variational iteration method [8], semi-inverse method [9].

In the case when a hyperbolic system of equations is reduced to a symmetric form, it seems possible to study the behavior of the system on discontinuous solutions, including those for problems with shock waves and contact discontinuities.

The exact solution of the discontinuity decay problem for heterophase media can lead to labor-consuming calculations; therefore, methods based on the approximate solution of the Riemann problem have become widespread.

In [10], the entropy analysis of schemes of this class was carried out, such as LxF (Lax-Friedrichs) [11], Rusanova [12], HLL (Harten, Lax, Leer) [13], Roe [14], EO (Engquist-Osher) [15], HLLC (Toro, Spruce, Speares) [16].

In [17], the HLL method was used to solve the Riemann problem about the decay of an arbitrary discontinuity for a system of differential equations

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of motion of a single-phase ideal fluid. To study the motion of two-phase media, the HLL method was used in [18–20].

In this paper, the Riemann problem of the discontinuity decay is studied for a two-phase medium under the assumption that the pressures and temperatures in the phases are equal. An approximate solution to a two-phase problem is verified by solving a single-phase problem.

2. Mathematical model

A heterophase medium is understood as a continuous medium, whose elementary volume consists of several (in the case in question, two) phases with different physical parameters.

The system of nonlinear equations of motion of a two-phase medium with one pressure was obtained in [21], following the general principles of the continuum mechanics. In the dissipative case, neglecting the processes of thermal conductivity and bulk viscosity, the system of equations has the form

$$\frac{\partial \rho_1}{\partial t} + \partial_i(\rho_1 u_{1,i}) = 0, \tag{2.1}$$

$$\frac{\partial \rho_2}{\partial t} + \partial_i(\rho_2 u_{2,i}) = 0. \tag{2.2}$$

$$\frac{\partial u_{1,i}}{\partial t} + u_{1,k}\partial_k u_{1,i} = -\frac{1}{\rho}\partial_i p - \frac{\rho_2}{2\rho}\partial_i(u_{1,k} - u_{2,k})^2 - b\frac{\rho_2}{\rho_1}(u_{1,i} - u_{2,i}) + \frac{\eta_1}{\rho_1}\Delta u_{1,i}. \tag{2.3}$$

$$\frac{\partial u_{2,i}}{\partial t} + u_{2,k}\partial_k u_{2,i} = -\frac{1}{\rho}\partial_i p + \frac{\rho_1}{2\rho}\partial_i(u_{1,k} - u_{2,k})^2 + b(u_{1,i} - u_{2,i}) + \frac{\eta_2}{\rho_2}\Delta u_{2,i}, \tag{2.4}$$

$$\frac{\partial S}{\partial t} + \partial_i\left(\frac{S}{\rho}(\rho_1 u_{1,i} + \rho_2 u_{2,i})\right) = \frac{R}{T}, \tag{2.5}$$

where ρ_1, ρ_2 are the partial densities of the phases of the two-phase medium, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ are the velocity fields of the phases of the two-phase medium, S is the entropy of a two-phase medium per unit volume, p is the pressure, T is the temperature, b is the interphase friction coefficient, η_1, η_2 are the shear viscosities of the phases of a two-phase medium. The dissipative function R has the form

$$R = b\rho_2(u_{1,i} - u_{2,i})^2 + \eta_1(\partial_k u_{1,i})^2 + \eta_2(\partial_k u_{2,i})^2.$$

The system of equations (2.1)–(2.5) is closed by the equations of state, which under the assumption of phase equilibrium with respect to pressure and temperature look like:

$$p = p(\rho, (\boldsymbol{u}_1 - \boldsymbol{u}_2)^2, S), \qquad T = T(\rho, (\boldsymbol{u}_1 - \boldsymbol{u}_2)^2, S).$$

As a consequence of system (2.1)–(2.5), we have the energy conservation law

$$\frac{\partial E}{\partial t} + \partial_k\left(\frac{p+E}{\rho}(\rho_1 u_{1,k} + \rho_2 u_{2,k}) + \frac{\rho_1\rho_2}{\rho} u_{1,i}(u_{1,i} - u_{2,i})(u_{1,k} - u_{2,k})\right) - \partial_k(\eta_{11} u_{1,i}\partial_k u_{1,i} + \eta_{22} u_{2,i}\partial_k u_{2,i}) = 0.$$

In the one-dimensional case let us reduce the system of two-velocity hydrodynamics (2.1)–(2.5) to a divergent form.

System (2.1)–(2.5) in the one-dimensional non-dissipative case takes the following form

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x}(\rho_1 u_1) = 0, \tag{2.6}$$

$$\frac{\partial \rho_2}{\partial t} + \frac{\partial}{\partial x}(\rho_2 u_2) = 0, \tag{2.7}$$

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\rho_2}{2\rho}\frac{\partial (u_1 - u_2)^2}{\partial x}, \tag{2.8}$$

$$\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\rho_1}{2\rho}\frac{\partial (u_1 - u_2)^2}{\partial x}, \tag{2.9}$$

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}\left(\frac{S}{\rho}(\rho_1 u_1 + \rho_2 u_2)\right) = 0. \tag{2.10}$$

Let us express the velocities in terms of the partial densities ρ_1, ρ_2, the total impulse $j = \rho_1 u_1 + \rho_2 u_2$ and the relative velocity $w = u_1 - u_2$

$$u_1 = \frac{j + \rho_2 w}{\rho}, \quad u_2 = \frac{j - \rho_1 w}{\rho}. \tag{2.11}$$

Taking into account these expressions, equations (2.6) and (2.7) take the following form

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x}\left(\rho_1\frac{j + \rho_2 w}{\rho}\right) = 0,$$

$$\frac{\partial \rho_2}{\partial t} + \frac{\partial}{\partial x}\left(\rho_2\frac{j - \rho_1 w}{\rho}\right) = 0.$$

Subtracting equation (2.9) from (2.8), we obtain the following equation for the relative velocity

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(w\frac{j + \rho_2 w}{\rho}\right) = 0.$$

Let us get the equation expressing the law of conservation of momentum. The sum of equation (2.6) multiplied by u_1 and equation (2.8) multiplied by ρ_1 is as follows

$$\frac{\partial}{\partial t}(\rho_1 u_1) + \frac{\partial}{\partial x}(\rho_1 u_1^2) + \frac{\rho_1}{\rho}\frac{\partial p}{\partial x} + \frac{\rho_1 \rho_2}{2\rho}\frac{\partial w^2}{\partial x} = 0. \tag{2.12}$$

The sum of equation (2.7) multiplied by u_2 and equation (2.9) multiplied by ρ_2 is

$$\frac{\partial}{\partial t}(\rho_2 u_2) + \frac{\partial}{\partial x}(\rho_2 u_2^2) + \frac{\rho_2}{\rho}\frac{\partial p}{\partial x} - \frac{\rho_1 \rho_2}{2\rho}\frac{\partial w^2}{\partial x} = 0. \tag{2.13}$$

Taking into account formulas (2.11), the sum of equations (2.12) and (2.13) gives the total momentum conservation law. Thus, we reduce the system of equations (2.6)–(2.10) to the divergent form

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x}\left(\rho_1 \frac{j + \rho_2 w}{\rho}\right) = 0, \tag{2.14}$$

$$\frac{\partial \rho_2}{\partial t} + \frac{\partial}{\partial x}\left(\rho_2 \frac{j - \rho_1 w}{\rho}\right) = 0, \tag{2.15}$$

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(w \frac{j + \rho_2 w}{\rho}\right) = 0, \tag{2.16}$$

$$\frac{\partial j}{\partial t} + \frac{\partial}{\partial x}\left(\frac{j^2}{\rho} + \frac{\rho_1 \rho_2 w^2}{\rho} + p\right) = 0, \tag{2.17}$$

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}\left(\frac{S}{\rho} j\right) = 0. \tag{2.18}$$

A consequence of the system of equations (2.14)–(2.18) is the energy conservation law

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left(\frac{p + E}{\rho} j + \frac{\rho_1 \rho_2}{\rho} u_1 (u_1 - u_2)^2\right) = 0. \tag{2.19}$$

3. The spectral analysis of the two-velocity hydrodynamics equations with one pressure in the one-dimensional case

Let us carry out the spectral analysis of system (2.6)–(2.10). We choose the equation of state in the form:

$$p = p(\rho, s), \tag{3.1}$$

where $s = S/\rho$ is the entropy of the two-phase medium per unit mass.

With the equation of state (3.1), the considered system has a representation in the form

$$\frac{\partial \rho_1}{\partial t} + \rho_1 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial \rho_1}{\partial x} = 0, \tag{3.2}$$

$$\frac{\partial \rho_2}{\partial t} + \rho_2 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial \rho_2}{\partial x} = 0, \tag{3.3}$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + \frac{p_\rho}{\rho}\left(\frac{\partial \rho_1}{\partial x} + \frac{\partial \rho_2}{\partial x}\right) + \frac{p_s}{\rho}\frac{\partial s}{\partial x} + \frac{\rho_2 w}{\rho}\left(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x}\right) = 0, \tag{3.4}$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + \frac{p_\rho}{\rho}\left(\frac{\partial \rho_1}{\partial x} + \frac{\partial \rho_2}{\partial x}\right) + \frac{p_s}{\rho}\frac{\partial s}{\partial x} - \frac{\rho_1 w}{\rho}\left(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x}\right) = 0, \tag{3.5}$$

$$\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} = 0, \tag{3.6}$$

where the following notation is used

$$p_\rho = \frac{\partial p}{\partial \rho}, \quad p_s = \frac{\partial p}{\partial s}, \quad u = \frac{\rho_1}{\rho} u_1 + \frac{\rho_2}{\rho} u_2.$$

We represent system (3.2)–(3.6) in the matrix form

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0,$$

where

$$U = (\rho_1, \rho_2, u_1, u_2, s)^T,$$

$$A = \begin{pmatrix} u_1 & 0 & \rho_1 & 0 & 0 \\ 0 & u_2 & 0 & \rho_2 & 0 \\ \frac{p_\rho}{\rho} & \frac{p_\rho}{\rho} & u_1 + \frac{\rho_2 w}{\rho} & -\frac{\rho_2 w}{\rho} & \frac{p_s}{\rho} \\ \frac{p_\rho}{\rho} & \frac{p_\rho}{\rho} & -\frac{\rho_1 w}{\rho} & u_2 + \frac{\rho_1 w}{\rho} & \frac{p_s}{\rho} \\ 0 & 0 & 0 & 0 & u \end{pmatrix},$$

Let us find the eigenvalues of the matrix A.

The equation $\|A - \lambda I\| = 0$ has the form

$$(u-\lambda)(u_1-\lambda)(u_2-\lambda)((u_2-\lambda)^2 + 2w(u_2-\lambda) + \rho^{-1}\rho_1 w^2 - p_\rho) -$$
$$2(u-\lambda)(u_1-\lambda)\rho^{-1}\rho_2 p_\rho w = 0.$$

Thus, $\lambda_1 = u$ and $\lambda_2 = u_1$ are the eigenvalues of the matrix A, the other eigenvalues satisfy the cubic equation

$$(u_2-\lambda)^3 + 2w(u_2-\lambda)^2 + (\rho^{-1}\rho_1 w^2 - p_\rho)(u_2-\lambda) - 2\rho^{-1}\rho_2 p_\rho w = 0.$$

Multiplying the resulting equation by $(\sqrt{p_\rho})^{-3}$, we obtain

$$\frac{(u_2-\lambda)^3}{(\sqrt{p_\rho})^3}+\frac{2w(u_2-\lambda)^2}{\sqrt{p_\rho}(\sqrt{p_\rho})^2}+\frac{(\rho^{-1}\rho_1 w^2-p_\rho)(u_2-\lambda)}{\sqrt{p_\rho}(\sqrt{p_\rho})^2}-\frac{2\rho^{-1}\rho_2 p_\rho w}{\sqrt{p_\rho}(\sqrt{p_\rho})^2}=0.$$

Let us change the variables

$$z=\frac{u_2-\lambda}{\sqrt{p_\rho}},\quad a=\frac{w}{\sqrt{p_\rho}}. \tag{3.7}$$

and obtain

$$z^3+2az^2-z(1-\rho^{-1}\rho_1 a^2)-2\rho^{-1}\rho_2 a=0. \tag{3.8}$$

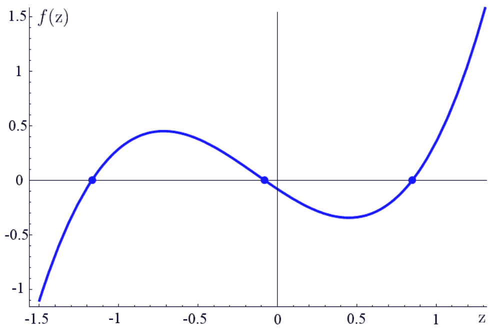

Figure 1

Figure 1 shows the graph of $f(z)$ for the values $\frac{u_1}{\sqrt{p_\rho}}=0.1$, $\frac{u_2}{\sqrt{p_\rho}}=0.3$, $\frac{\rho_1}{\rho}=0.8$ and $\frac{\rho_2}{\rho}=0.2$. The function $f(z)$ has three real roots.

For small values of the parameter a, we find the roots of the equation $f(z)$ by expanding $z(a)$ into a series in powers of a. Let

$$z(a)=z_0+ab. \tag{3.9}$$

Substituting expression (3.9) into equation (3.8), assuming that a is sufficiently small, we arrive at the following equation

$$a^0(z_0^3-z_0)+a^1(3z_0^2b+z_0^2-b-2\rho^{-1}\rho_2)=0.$$

The identity is attained if the coefficients of the powers of a are equal to zero. The coefficient for a^0 is zero for the following values $z_{0,1}=0$, $z_{0,2}=1$, $z_{0,3}=-1$. Accordingly, we obtain the conditions for the coefficient being equal to zero of at a^1:

$$b_1=-2\rho^{-1}\rho_2,\quad b_2=b_3=\rho^{-1}\rho_2-1.$$

Substituting these conditions into (3.9), we obtain

$$z_1(a)=-2\rho^{-1}\rho_2 a,\quad z_2(a)=1-\rho^{-1}\rho_1 a,\quad z_3(a)=-1-\rho^{-1}\rho_1 a.$$

From (3.7) we express $\lambda = u_2 - z\sqrt{p_\rho}$. Thus, we have obtained the eigenvalues of the matrix A:

$$\lambda_1 = u, \tag{3.10}$$

$$\lambda_2 = u_1, \tag{3.11}$$

$$\lambda_3 = u_2 + 2\rho^{-1}\rho_2 w, \tag{3.12}$$

$$\lambda_4 = u_2 - \sqrt{p_\rho} + \rho^{-1}\rho_1 w, \tag{3.13}$$

$$\lambda_5 = u_2 + \sqrt{p_\rho} + \rho^{-1}\rho_1 w, \tag{3.14}$$

which will be used below to find the values of the numerical fluxes.

4. Numerical results

The system of equations of the two-velocity hydrodynamics with phase equilibrium in terms of pressure (2.14)–(2.17), (2.19) has a representation in the form of discrete analogs of the conservation laws

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \tag{4.1}$$

where

$$U = (\rho_1, \rho_2, w, j, E)^T,$$

$$F(U) = \begin{pmatrix} \rho_1 \dfrac{j + \rho_2 w}{\rho} \\ \rho_2 \dfrac{j - \rho_1 w}{\rho} \\ w\dfrac{j + \rho_2 w}{\rho} \\ \dfrac{j^2}{\rho} + \dfrac{\rho_1 \rho_2 w^2}{\rho} + p \\ \dfrac{j}{\rho}(p + E) + \dfrac{\rho_1 \rho_2}{\rho} u_1 (u_1 - u_2)^2 \end{pmatrix}.$$

We will seek a numerical solution to system (2.14)–(2.17), (2.19) in the computational domain $\Omega = \{(t, x) :\ 0 \le t < T,\ 0 \le x \le 1\}$, with the following boundary conditions

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = \left.\frac{\partial U}{\partial x}\right|_{x=1} = 0.$$

For integration with respect to time, we will use a first-order scheme in the space-time cell $[x_{i-1/2}, x_{i+1/2}] \times [t^n, t^{n+1}]$ with the steps $\Delta x = x_{i+1/2} - x_{i-1/2}$ and $\Delta t = t^{n+1} - t^n$:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}), \tag{4.2}$$

where U_i^n is the approximation of U in the cell x_i at the time instant t^n, $\mathcal{F}_{i\pm 1/2}$ are the numerical flows at the borders of cells, the indices running over the values $i = \overline{0, I}$, $n = \overline{0, N}$.

Knowing the formulas for calculating the characteristics (3.10)–(3.14) allows us to apply the HLL method for the system of equations (2.14)–(2.17), (2.19) [13]:

$$\mathcal{F}_{i+\frac{1}{2}} = \frac{(a_R^- - a_L^-)F_R + (a_R^+ - a_L^+)F_L - \frac{1}{2}(a_R|a_L| - a_L|a_R|)(U_R - U_L)}{a_R - a_L},$$

$$a^+ = \max(a, 0) = \frac{a + |a|}{2}, \quad a^- = \min(a, 0) = \frac{a - |a|}{2},$$

where a_L, a_R are the lower and the upper boundaries, respectively, for the lowest and the highest wave velocities. The index L refers to the ith cell and R refers to the $(i+1)$th. The time step at the nth time layer is calculated by the formula

$$\Delta t = \frac{K \Delta x}{a_{\max}^n},$$

where K is the Courant number, $a_{\max}^n$ is the maximum velocity of the longitudinal sound wave at a given time level.

The HLL method was used to solve the Riemann problem about the decay of an arbitrary discontinuity for the system of equations of motion of a single-phase ideal fluid [17], which looks like

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0, \tag{4.3}$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} = 0, \tag{4.4}$$

$$\frac{\partial E}{\partial t} + \frac{\partial (u(E + p))}{\partial x} = 0. \tag{4.5}$$

Here u is the velocity, ρ is the density, p is the pressure, $E = \rho(u^2/2 + e)$ is the total energy of an ideal fluid.

The specific internal energy e is given in the form

$$e = e(\rho, p) = \frac{p}{(\gamma - 1)\rho}. \tag{4.6}$$

The HLL method in [17] was verified on test problems in comparison with the solutions obtained using Godunov's scheme. Consider such two test problems with the following initial data presented in the table, where x_0 is a break point.

Test	ρ_L	u_L	p_L	ρ_R	u_R	p_R	x_0
1	1.0	0.75	1.0	0.125	0.0	0.1	0.3
2	1.0	-2	0.4	1.0	2.0	0.4	0.5

Test 1— "Sod test problem" [22]. In this test, jumps in pressure, density, and velocity are set as initial conditions. For a two-phase medium, the values of the initial data in the phases are taken to be the same. The solution is a left rarefaction wave, a contact discontinuity, and a right shock wave.

Test 2— "123 problem" [23]. In this test, the speed jump is set as the initial conditions. The velocities of the phases of a two-phase medium are locally taken to be equal. The solution consists of two strong rarefactions and a trivial continuous contact discontinuity.

To compare the numerical solutions in these tests for system (2.14)–(2.17), (2.19), let us make the following assumption about the equation of state phases: the adiabatic exponents γ of the phases are taken to be the same and equal 1.4. In this case, the internal energy e of the two-phase medium is a function of the hydrodynamic pressure of the two-phase medium and is expressed in the form of (4.6). Also, to compare a two-phase medium with a single-phase liquid, the parameters of each of the phases of the two-phase medium were chosen to be the same and corresponding to the parameters of an ideal fluid.

Figures 2 and 3 show a comparison of the solutions of test problems 1 and 2 for a single-phase ideal fluid and a two-phase medium with equal values

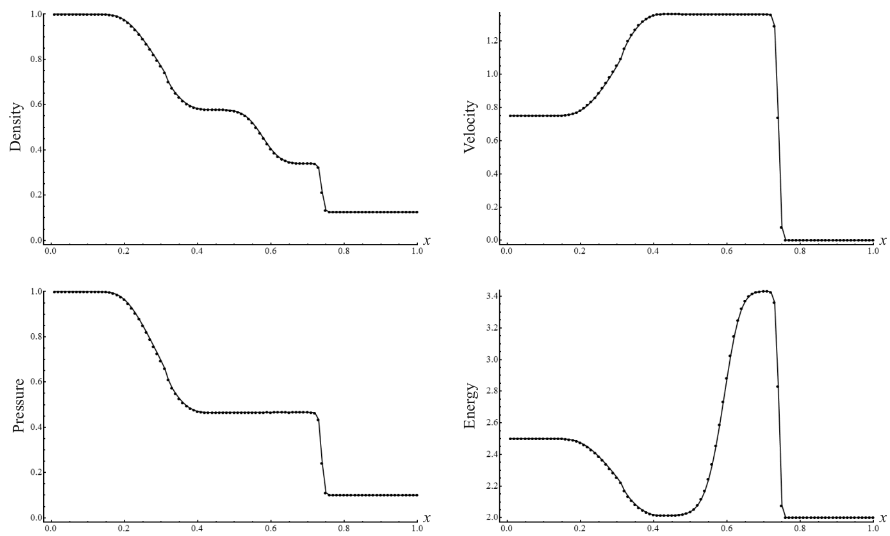

Figure 2. Test 1. Calculation results for an ideal fluid (solid line) and for a two-phase liquids (points). The time moment is 0.2

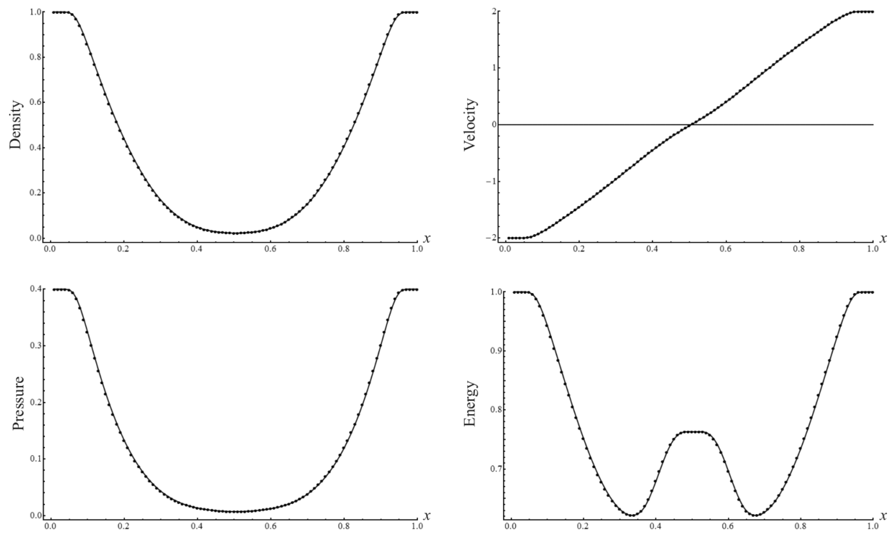

Figure 3. Test 2. Calculation results for an ideal fluid (solid line) and for a two-phase liquids (points). The time moment is 0.15

of the phase parameters. The profiles of density, velocity, pressure, and a specific internal energy are presented.

These figures demonstrate a good agreement between the solution to the problem of discontinuity decay in a two-phase medium (with the same phase parameters) with the solutions for the single-phase liquid.

In the case of setting different values of the partial densities and velocities of the phases of the two-phase medium in test 1, there is a difference in the behavior of the phases. Figure 4 shows the profiles of the densities and velocities of the phases of a two-phase medium. The test sets the following initial data: the values of the partial densities and velocities of the phases to the left of the discontinuity $\rho_{1L} = 0.8$, $\rho_{2L} = 0.2$, $v_{1L} = 0.65$, $v_{2L} = 1.15$ and to the right of the gap $\rho_{1R} = 0.9$, $\rho_{2R} = 0.1$, $v_{1R} = v_{2R} = 0$; the pressure values are considered equal and correspond to the table above. With this

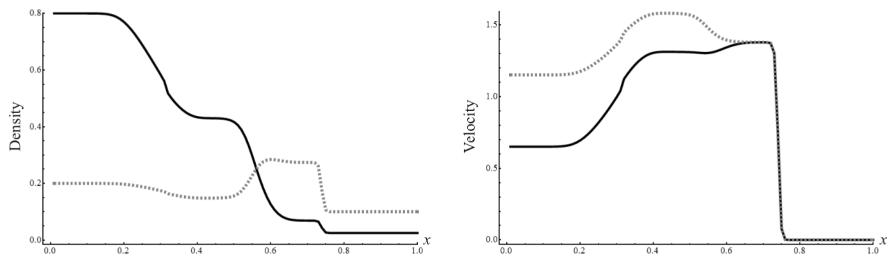

Figure 4. Calculation results for the first phase (solid line) and for the second one (dashed line). The time moment is 0.2

setting of the initial data, the total density and average velocity of the two-phase medium correspond to the values presented in the table above.

Figure 4 demonstrates the appearance of differences in the values of densities and phase velocities. Moreover, the characteristic behavior of the total density and average velocity of a two-phase medium coincides with a single-phase analogue.

Conclusion

The paper considers a system of equations of motion of a two-phase medium with phase equilibrium in terms of pressure and temperature. In the one-dimensional dissipation-free case, the system is reduced to a divergent form. The characteristic values are found, which make it possible to apply the numerical HLL method to study the behavior of a two-phase system on the example of the problem of discontinuity decay. Comparison of the obtained approximate solutions with the equations of motion of an ideal fluid shows their good agreement.

References

[1] Hirota R. Exact solution of the Korteweg-de Vries equation for multiple collisions of olitons // Phys. Rev. Lett. — 1971. — Vol. 27. — P. 1192–1194.

[2] Ablowitz M.J., Ablowitz M.A., Clarkson P.A., Clarkson Peter A. Solitons, Nonlinear Evolution Equations and Inverse Scattering. — Cambridge: Cambridge University Press, 1991.

[3] Rogers C., Shadwick W.F. Backlund Transformations and Their Applications. — New York: Academic Press, 1982.

[4] Matveev V.A., Salle M.A. Darboux Transformations and Solitons. — Springer, 1991.

[5] Cariello F., Tabor M. Similarity reductions from extended Painleve expansions for nonintegrable evolution equations // Physica D. — 1991. — Vol. 53. — P. 59–70.

[6] Lagno V.I., Spichak S.V., Stogniy V.I. Symmetry Analysis of Equations of Evolutionary Type. — Izhevsk: Institute of Computer Research, 2004 (In Russian).

[7] Wang M.L., Zhou Y.B., Li Z.B. Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics // Phys. Lett. A. — 1996. — Vol. 216. — P. 67–75.

[8] He J.H. Variational iteration method — a kind of non-linear analytical technique: some examples // Int. J. Non-Linear Mech. — 1999. — Vol. 34. — P. 699–708.

[9] He J.H. Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbo machinery aerodynamics // Int. J. Turbo Jet-Engines. — 1997. — Vol. 14, No. 1. — P. 23–28.

[10] Lipnitskiy Yu.M., Safronov A.V. Modification of Godunov's numerical method for the equations of gas dynamics // Uchenye Zapiski TsAGI. — 2012. — Vol. XLIII, No. 4. — P. 20–29 (In Russian).

[11] Lax P.D. Weak solutions of nonlinear hyperbolic equations and their numerical computation // Communications on Pure and Applied Mathematics. — 1954. — Vol. 7. — P. 159–193.

[12] Rusanov V.V. Calculation of the interaction of nonstationary shock waves with obstacles // ZhVM and MF. — 1961. — Vol. 1, No. 2. — P. 267–279.

[13] Harten A., Lax P.D., Leer B.V. On upstream differencing and Godunov-type schemes for hyperbolic conservation law // SIAM Review. — 1983. — Vol. 25, No. 1. — P. 35–61.

[14] Roe P.L. Approximate Riemann solvers, parameter vectors, and difference schemes // J. Comput. Phis. — 1981. — Vol. 43, No. 2. — P. 357–372.

[15] Enguist B., Osher S. One-sided difference approximation for nonlinear conservation laws // Math. Comp. — 1981. — Vol. 3. — P. 321–351.

[16] Toro E.F., Spruce M., Speares S. Restoration of the contact surface in the HLL Riemann solver // Shock Saves. — 1994. — Vol. 4. — P. 25–34.

[17] Toro E.F. Riemann Solvers and Numerical Methods Fluid Dynamics. — Springer, 2009.

[18] Ivanov I.E., Kryukov I.A. Numerical algorithm for modeling two-phase flows containing phase boundaries // Physicochemical Kinetics in Gas Dynamics. — 2012. — Vol. 13, No. 4. — P. 3 (In Russian).

[19] Tukhvatullina R.R., Frolov S.M. Correctness of Euler's non-isothermal model for two-phase flows // Combustion and Explosion. — 2016. — Vol. 9, No. 4. — P. 36–46 (In Russian).

[20] Utkin P.S. Mathematical modeling of the interaction of a shock wave with a dense filling of particles in the framework of a two-fluid approach // Chemical Physics. — 2017. — Vol. 36, No. 11. — P. 61–71 (In Russian).

[21] Dorovsky V.N., Perepechko Yu.V. Theory of partial melting // Geology and Geophysics. — 1989. — No. 9. — P. 56–64 (In Russian).

[22] Sod G.A. A survey of several finite difference methods for systems of nonlinear hyperbolic conservation laws // J. Comput. Phys. — 1978. — Vol. 27. — P. 1–31.

[23] Einfeldt B., Munz C.D., Roe P.L., Sjogreen B. On Godunov-type methods near low densities // J. Comput. Phys. — 1991. — Vol. 92. — P. 273–295.