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Two-phase compressible flow in a rectangular channel*

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Abstract. The paper investigates a mathematical model of a two-phase medium under the assumption that the phases are in equilibrium with respect to pressure and temperature. Dissipative effects are determined by interfacial friction. In the one-dimensional case, a group analysis of the dissipative system of equations for a two-phase medium is carried out. In the two-dimensional case, the system is solved numerically by the third-order TVD Runge-Kutta method. A problem of two-phase medium motion in a channel in a gravitational field is solved.

Keywords: heterophase medium, group analysis, TVD Runge-Kutta method, channel flow.

Introduction

Modeling the evolution of natural systems, such as magmatic and fluid-magmatic systems, is possible due to general non-stationary nonlinear models that describe the dynamics of heat and mass transfer of heterophase multicomponent media over a wide range of time and space scales. The choice of the physical model parameters corresponding to a real system is an urgent task that requires an analytical and numerical study of the equations taken for the mathematical model. This article provides an analytical and numerical analysis of the equations for two-phase medium flow obtained by a phenomenological method that ensures their thermodynamic correctness. The symmetry properties of the equations are studied by the group analysis method, which, in turn, is based on the study of differential equations invariance for one of the parameters from the Lie groups of point transformations [1–6]. Group analysis can unify the analytic methods for constructing explicit solutions of differential equations, especially for the case of nonlinear partial differential equations. Nonlinear systems of equations for two-velocity medium mechanics (without temperature effects) was studied in [7] on the basis of group analysis method. This article considers a complete system of equations, including the energy conservation law along with a more general equation of state [8]. The compressible two-phase media flow problem in this work has been solved numerically. The choice of the

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third-order TVD Runge-Kutta method is determined by a need in considering the acoustic phenomena at the background of media fluid motion.

1. System of dissipative equations for a two-phase medium

A system of equations with two-velocity hydrodynamics approach with account for phase equilibrium and dissipation (caused only by the force of interfacial friction [8]) takes the form:

$$\frac{\partial}{\partial t}\rho_1 + \operatorname{div}(\rho_1 \boldsymbol{u}_1) = 0, \quad \frac{\partial}{\partial t}\rho_2 + \operatorname{div}(\rho_2 \boldsymbol{u}_2) = 0, \tag{1.1}$$

$$\frac{\partial}{\partial t}\boldsymbol{u}_1 + (\boldsymbol{u}_1, \nabla)\boldsymbol{u}_1 + \frac{1}{\rho}\nabla p + \frac{\rho_2}{2\rho}\nabla w^2 + b\frac{\rho_2}{\rho_1}\boldsymbol{w} = 0, \tag{1.2}$$

$$\frac{\partial}{\partial t}\boldsymbol{u}_2 + (\boldsymbol{u}_2, \nabla)\boldsymbol{u}_2 + \frac{1}{\rho}\nabla p - \frac{\rho_1}{2\rho}\nabla w^2 - b\boldsymbol{w} = 0, \tag{1.3}$$

$$\frac{\partial}{\partial t}E + \operatorname{div}\Big((p+E)\Big(\frac{\rho_1}{\rho}\boldsymbol{u}_1 + \frac{\rho_2}{\rho}\boldsymbol{u}_2\Big) + \frac{\rho_1\rho_2}{\rho}\boldsymbol{u}_1 w^2\Big) = 0, \tag{1.4}$$

where

$$E = E_0 + \rho_1(\boldsymbol{u}_2, \boldsymbol{w}) + \frac{1}{2}\rho u_2^2, \quad p = p(\rho, w^2, E_0), \quad \boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2, \quad \rho = \rho_1 + \rho_2.$$

The equation of state for a composite two-velocity medium is taken as

$$e_n(\rho_n) = p_{0n}\frac{1}{\rho_{0n}^2}(\rho_n - \rho_{0n}) - \frac{1}{2k_n\rho_{0n}^3}(\rho_n - \rho_{0n})^2,$$

and the pressure in each phase is determined by the formula

$$p_n(\rho_n) = p_{0n} + \frac{1}{k_n}\Big(\frac{\rho_n - \rho_{0n}}{\rho_{0n}}\Big),$$

where ρ_{0n} is the phase density under normal conditions, $p_{0n} = p_n(\rho_{0n}, 0)$ is the pressure under normal conditions, and k_n is the bulk modulus.

2. Invariant solutions in one-dimensional case

The group properties of the equations of two-velocity hydrodynamics are studied for 1D systems according to the following algorithm [1–6]: specifying the form of the infinitesimal operator allowed by the system; construction of the first continuation of the infinitesimal operator; action by the continued operator on each equation of the system; transition to a differential manifold; fulfillment of the splitting of the invariance conditions; solving the defining equations system; finding the optimal systems of subalgebras; normalizing the optimal systems of subalgebras; construction of invariant and partially invariant submodels of the system; and, finally, finding the invariant and partially invariant solutions.

Let us introduce the notation

$$\varphi = \rho_1, \quad \psi = \rho_2, \quad u = u_1, \quad v = u_2, \quad \omega = w^2.$$

Let us express the derivative of function $p(\rho, \omega, E_0)$ with respect to x through the functions of its arguments, and reduce system (1.1)–(1.4) to one-dimensional case in the form

$$\varphi_t + \varphi_x u + u_x \varphi = 0, \quad \psi_t + \psi_x v + v_x \psi = 0, \tag{2.1}$$

$$u_t + u u_x + \rho^{-1}\big(p_\rho(\varphi_x + \psi_x) + (2p_\omega + \psi) w w_x + p_{E_0} E_{0x}\big) + \varphi^{-1} b \psi w = 0, \tag{2.2}$$

$$v_t + v v_x + \rho^{-1}\big(p_\rho(\varphi_x + \psi_x) + w w_x (2p_\omega - \varphi) + p_{E_0} E_{0x}\big) - bw = 0. \tag{2.3}$$

$$E_{0t} + \rho^{-1} j E_{0x} + b\varphi\omega + \rho^{-2}\big(\psi w (p + E_0) - \psi^2 w^3\big)\varphi_x - \\ \rho^{-2}\big(\varphi w (p + E_0) + \varphi^2 w^3\big)\psi_x + \rho^{-1}\big(\varphi(p + E_0) + \varphi^2\omega + 3\varphi\psi\omega\big)u_x + \\ \rho^{-1}(\psi(p + E_0) - 2\varphi\psi\omega)v_x = 0. \tag{2.4}$$

The infinitesimal operator admitted by system (2.1)–(2.4) has the form

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial \varphi} + \beta\frac{\partial}{\partial \psi} + \delta\frac{\partial}{\partial u} + \gamma\frac{\partial}{\partial v} + \kappa\frac{\partial}{\partial E_0}, \tag{2.5}$$

where the functions τ, ξ, α, β, δ, γ, κ depend on the variables t, x, φ, ψ, u, v, E_0.

Let us construct the first continuation of the operator X

$$\bar{X} = X + \Phi^t\frac{\partial}{\partial \varphi_t} + \Phi^x\frac{\partial}{\partial \varphi_x} + \Psi^t\frac{\partial}{\partial \psi_t} + \Psi^x\frac{\partial}{\partial \psi_x} + \\ U^t\frac{\partial}{\partial u_t} + U^x\frac{\partial}{\partial u_x} + V^t\frac{\partial}{\partial v_t} + V^x\frac{\partial}{\partial v_x} + E^t\frac{\partial}{\partial E_{0t}} + E^x\frac{\partial}{\partial E_{0x}},$$

where the coefficients Φ^t, Φ^x, Ψ^t, Ψ^x, U^t, U^x, V^t, V^x, E^t, E^x depend on t, x, φ, ψ, u, v, E_0, φ_t, φ_x, ψ_t, ψ_x, u_t, u_x, v_t, v_x, E_{0t}, E_{0x}. Let us act as an operator $\bar{X}$ on equations (2.1)–(2.3), considering the latter as differential manifolds in independent variables t, x, φ, ψ, u, v, E_0, φ_t, φ_x, ψ_t, ψ_x, u_t, u_x, v_t, v_x, E_{0t}, E_{0x}:

$$\Phi^t + \delta\varphi_x + \Phi^x u + \alpha u_x + U^x\varphi = 0, \tag{2.6}$$

$$\Psi^t + \gamma\psi_x + \Psi^x v + \beta v_x + V^x\psi = 0, \tag{2.7}$$

$$U^t + \delta u_x + U^x u - \rho^{-2}(p_\rho(\varphi_x + \psi_x) + 2p_\omega w(u_x - v_x) + p_{E_0}E_{0x})(\alpha + \beta) +$$
$$\rho^{-1}(p_{\rho\rho}(\varphi_x + \psi_x) + 2p_{\omega\rho}w(u_x - v_x) + p_{\rho E_0}E_{0x})(\alpha + \beta) +$$
$$\rho^{-1}(2p_{\rho\omega}w(\varphi_x + \psi_x) + 4p_{\omega\omega}w^2(u_x - v_x) + 2p_{\omega E_0}wE_{0x})(\delta - \gamma) +$$
$$\rho^{-1}(p_{\rho E_0}(\varphi_x + \psi_x) + 2p_{\omega E_0}w(u_x - v_x) + p_{E_0E_0}E_{0x})\kappa +$$
$$2\rho^{-1}p_\omega w(U^x - V^x) - \rho^{-2}\psi w(u_x - v_x)(\alpha + \beta) + \rho^{-1}w(u_x - v_x)\beta +$$
$$\rho^{-1}(\psi(u_x - v_x)(\delta - \gamma) + p_\rho(\Phi^x + \Psi^x) + \psi w(U^x - V^x) + p_{E_0}E^x) +$$
$$2\rho^{-1}p_\omega(u_x - v_x)(\delta - \gamma) + \varphi^{-1}bw\beta - \varphi^{-2}\psi bw\alpha + \varphi^{-1}\psi b(\delta - \gamma) = 0, \quad (2.8)$$

$$V^t + \gamma v_x + V^x v - \rho^{-2}(p_\rho(\varphi_x + \psi_x) + 2p_\omega w(u_x - v_x) + p_{E_0}E_{0x})(\alpha + \beta) +$$
$$\rho^{-1}(p_{\rho\rho}(\varphi_x + \psi_x) + 2p_{\omega\rho}w(u_x - v_x) + p_{\rho E_0}E_{0x})(\alpha + \beta) +$$
$$\rho^{-1}(2p_{\rho\omega}w(\varphi_x + \psi_x) + 4p_{\omega\omega}w^2(u_x - v_x) + 2p_{\omega E_0}wE_{0x})(\delta - \gamma) +$$
$$\rho^{-1}(p_{\rho E_0}(\varphi_x + \psi_x) + 2p_{\omega E_0}w(u_x - v_x) + p_{E_0E_0}E_x)\kappa +$$
$$2\rho^{-1}p_\omega w(U^x - V^x) + \rho^{-2}\varphi w(u_x - v_x)(\alpha + \beta) - \rho^{-1}w(u_x - v_x)\alpha -$$
$$\rho^{-1}(\varphi(u_x - v_x)(\delta - \gamma) - p_\rho(\Phi^x + \Psi^x) + \varphi w(U^x - V^x) - p_{E_0}E^x) +$$
$$2\rho^{-1}p_\omega(u_x - v_x)(\delta - \gamma) - b(\delta - \gamma) = 0. \quad (2.9)$$

After substituting the continuation formulas and replacing the time derivatives with derivatives in respect of the remaining quantities into equations (2.6), (2.7), we obtain the equations that are nonhomogeneous quadratic forms for free variables φ_x, ψ_x, u_x, v_x, E_{0x}. Having performed the splitting of the invariance conditions, assuming that $p_\rho \neq 0$, we obtain the condition that $\tau = \tau(t,x)$, $\xi = \xi(t,x)$. Performing a similar procedure for equations (2.8), (2.9), considering the function p and its derivatives as additional free variables, provided that $u \neq v$, we arrive at the following solution of the system of constitutive relations:

$$\tau = \text{const}, \quad \xi = \xi(t), \quad \alpha = \beta = \kappa = 0, \quad \xi_t = \delta = \gamma = \text{const}.$$

Equation (2.5), after substituting the continuation formulas and changing the time derivatives, does not entail additional conditions on the coefficients of infinitesimal operator and reduces to a trivial equality. Thus, we have a solution:

$$\tau = A, \quad \xi = Ct + B, \quad \delta = \gamma = C.$$

By choosing one of the constants (A, B, C) from of the solution for a system defining equations and take it equal to one, then the rest constants are taken zero, we obtain the basis for the kernel of the main Lie algebra for equations (2.1)–(2.4):

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}.$$

The assumption that the function is not constant is not essential. All groups generated by operators under this assumption are also admissible.

The optimal systems Θ_s for Lie algebra L_3 are presented in the following table:

r	i	Basis	Normalizer
1	1	X_1	$\langle X_1, X_2, X_3\rangle$
	2	$X_2 + cX_3$	$\langle X_1, X_2 + cX_3\rangle$; $\langle X_1, X_2, X_3\rangle$ $(c = 1)$
	3	X_3	$\langle X_1, X_3\rangle$
2	1	$\langle X_1, X_3\rangle$	$\langle X_1, X_2, X_3\rangle$
	2	$\langle X_1, X_2 + cX_3\rangle$	$\langle X_1, X_2, X_3\rangle$
3	1	$\langle X_1, X_2, X_3\rangle$	$= \langle X_1, X_2, X_3\rangle$

Let us find the invariant solutions for all non-similar subgroups whose algebras are included in optimal systems.

Submodel 1.1. The operator $X_1 = \dfrac{\partial}{\partial x}$ has invariants

$$J_1 = t, \quad J_2 = \rho_1, \quad J_3 = \rho_2, \quad J_4 = u_1, \quad J_5 = u_2, \quad J_6 = E_0.$$

The invariant solution is sought in the form

$$\rho_1 = \rho_1(t), \quad \rho_2 = \rho_2(t), \quad u_1 = u_1(t), \quad u_2 = u_2(t), \quad E_0 = E_0(t).$$

Substituting the desired functions into equations (2.1)–(2.4) will give the following

$$\frac{\partial}{\partial t}\rho_1 = 0, \quad \frac{\partial}{\partial t}\rho_2 = 0, \tag{2.10}$$

$$\frac{\partial}{\partial t}u_1 + b\frac{\rho_2}{\rho_1}(u_1 - u_2) = 0, \quad \frac{\partial}{\partial t}u_2 - b(u_1 - u_2) = 0, \tag{2.11}$$

$$\frac{\partial}{\partial t}\Big(E_0 + \rho_1 u_2(u_1 - u_2) + \frac{1}{2}\rho u_2^2\Big) = 0.$$

From equations (2.10) we have $\rho_1 = c_1$, $\rho_2 = c_2$.

Multiplying the first equality from (2.11) by $\frac{\rho_1}{\rho_2}$ and adding the second one, we obtain

$$\frac{\partial}{\partial t}\Big(u_2 + \frac{c_1}{c_2}u_1\Big) = 0, \quad \text{hence } u_2 + \frac{c_1}{c_2}u_1 = c_3.$$

Then, substituting $u_2 = -\frac{c_1}{c_2}u_1 + c_3$ into the difference of the first and the second equalities from (2.11), we derive

$$\frac{\partial}{\partial t}u_1 + b\frac{c_1+c_2}{c_1}u_1 = b\frac{c_2c_3}{c_1}.$$

Therefore,

$$u_1 = c_4\exp\Big(-b\frac{c_1+c_2}{c_1}t\Big) + \frac{c_2c_3}{c_1+c_2}.$$

Thus, we arrive at invariant solution

$$\rho_1 = c_1, \quad \rho_2 = c_2,$$
$$u_1 = c_4\exp\Big(-b\frac{c_1+c_2}{c_1}t\Big) + \frac{c_2c_3}{c_1+c_2},$$
$$u_2 = -\frac{c_1c_4}{c_2}\exp\Big(-b\frac{c_1+c_2}{c_1}t\Big) + \frac{c_2c_3}{c_1+c_2},$$
$$E_0 = c_5 - c_1u_2(u_1-u_2) - \frac{1}{2}(c_1+c_2)u_2^2.$$

Submodel 1.2a. The operator $X_2 = \dfrac{\partial}{\partial t}$ has invariants

$$J_1 = x, \quad J_2 = \rho_1, \quad J_3 = \rho_2, \quad J_4 = u_1, \quad J_5 = u_2, \quad J_6 = E_0.$$

The solution is sought in the form

$$\rho_1 = \rho_1(x), \quad \rho_2 = \rho_2(x), \quad u_1 = u_1(x), \quad u_2 = u_2(x), \quad E_0 = E_0(x).$$

After substituting the required functions into equations (2.1), we obtain, $\dfrac{\partial\rho_1u_1}{\partial x} = 0$, $\dfrac{\partial\rho_2u_2}{\partial x} = 0$, therefore, $\rho_1u_1 = c_1$, $\rho_2u_2 = c_2$. Then equations (2.2)–(2.4) take the form

$$u_1\frac{\partial u_1}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\rho_2}{2\rho}\frac{\partial}{\partial x}(u_1-u_2)^2 - b\frac{\rho_2}{\rho_1}(u_1-u_2),$$
$$u_2\frac{\partial u_2}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\rho_1}{2\rho}\frac{\partial}{\partial x}(u_1-u_2)^2 + b(u_1-u_2),$$
$$\frac{\partial}{\partial x}\Big(\frac{p+E}{\rho}(\rho_1u_1+\rho_2u_2) + \frac{\rho_1\rho_2}{\rho}u_1(u_1-u_2)^2\Big) = 0.$$

The invariant solutions depend on the specification of the function p.

Submodel 1.2b. The operator $X_2 + cX_3 = \dfrac{\partial}{\partial t} + ct\dfrac{\partial}{\partial x} + c\dfrac{\partial}{\partial u_1} + c\dfrac{\partial}{\partial u_2}$ has invariants

$$J_1 = \frac{ct^2}{2} - x, J_2 = \rho_1, J_3 = \rho_2, J_4 = u_1 - ct, J_5 = u_2 - ct, J_6 = E_0.$$

The solution is sought in the form

$$\rho_1 = \rho_1(y), \quad \rho_2 = \rho_2(y), \quad u_1 = u_1(y)+ct, \quad u_2 = u_2(y)+ct, \quad E_0 = E_0(y),$$

where $y = \frac{1}{2}ct^2 - x$.

Substituting the desired functions into equations (2.1) we obtain

$$-\frac{\partial \rho_1 u_1}{\partial y} = 0, \quad -\frac{\partial \rho_2 u_2}{\partial y} = 0.$$

Therefore, $\rho_1 u_1 = c_1$, $\rho_2 u_2 = c_2$. Then equations (2.2)–(2.4) take the form

$$\begin{aligned}
&c - u_1 \frac{\partial u_1}{\partial y} = \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\rho_2}{2\rho}\frac{\partial}{\partial y}(u_1 - u_2)^2 - b\frac{\rho_2}{\rho_1}(u_1 - u_2),\\
&c - u_2 \frac{\partial u_2}{\partial y} = \frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{\rho_1}{2\rho}\frac{\partial}{\partial y}(u_1 - u_2)^2 + b(u_1 - u_2),\\
&\frac{\partial}{\partial y}\Big(\frac{p+E}{\rho}(\rho_1 u_1 + \rho_2 u_2) + \frac{\rho_1\rho_2}{\rho}u_1(u_1 - u_2)^2\Big) = 0.
\end{aligned}$$

The invariant solutions depend on the specification of the function p.

Submodel 1.3. The operator $X_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}$ has invariants

$$J_1 = t, \quad J_2 = \rho_1, \quad J_3 = \rho_2, \quad J_4 = u_1 - \frac{x}{t}, \quad J_5 = u_2 - \frac{x}{t}, \quad J_6 = E_0.$$

The solution is sought in the form

$$\rho_1 = \rho_1(t), \quad \rho_2 = \rho_2(t), \quad u_1 = u_1(t) + \frac{x}{t}, \quad u_2 = u_2(t) + \frac{x}{t}, \quad E_0 = E_0(t).$$

Substituting the desired functions into equations (2.1)–(2.4) will give the following

$$\frac{\partial \rho_1}{\partial t} + \frac{\rho_1}{t} = 0, \quad \frac{\partial \rho_2}{\partial t} + \frac{\rho_2}{t} = 0, \tag{2.12}$$

$$\frac{\partial}{\partial t}u_1 + \frac{u_1}{t} + b\frac{\rho_2}{\rho_1}(u_1 - u_2) = 0, \quad \frac{\partial}{\partial t}u_2 + \frac{u_2}{t} - b(u_1 - u_2) = 0, \tag{2.13}$$

$$\frac{\partial}{\partial t}E_0 + \frac{1}{t}\Big(p + E_0 + \rho_1(1+tb)(u_1 - u_2)^2\Big) = 0.$$

From equations (2.12) we have $\rho_1 = c_1/t$, $\rho_2 = c_2/t$. Transforming equations (2.13) the same way as for (2.11), we obtain

$$\frac{\partial}{\partial t}\Big(u_2 + \frac{c_1}{c_2}u_1\Big) + \frac{1}{t}\Big(u_2 + \frac{c_1}{c_2}u_1\Big) = 0, \quad \frac{\partial}{\partial t}u_1 + \Big(\frac{1}{t} + b\frac{c_1 + c_2}{c_1}\Big)u_1 = b\frac{c_2 c_3}{c_1 t}.$$

The following expressions will be the solution of these equations

$$u_2 + \frac{c_1}{c_2} u_1 = \frac{c_3}{t}, \quad u_1 = \frac{c_4}{t} \exp\Big(-b\frac{c_1+c_2}{c_1} t\Big) + \frac{c_2 c_3}{(c_1+c_2)t}.$$

Thus, we arrive at the solution

$$\rho_1 = \frac{c_1}{t}, \quad \rho_2 = \frac{c_2}{t},$$
$$u_1 = \frac{c_4}{t} \exp\Big(-b\frac{c_1+c_2}{c_1} t\Big) + \frac{c_2 c_3}{(c_1+c_2)t} + \frac{x}{t},$$
$$u_2 = -\frac{c_1 c_4}{c_2 t} \exp\Big(-b\frac{c_1+c_2}{c_1} t\Big) + \frac{c_2 c_3}{(c_1+c_2)t} + \frac{x}{t},$$
$$\frac{\partial}{\partial t} E_0 + \frac{1}{t}\Big(p + E_0 + \frac{c_1}{t}(1+tb)(u_1-u_2)^2\Big) = 0.$$

The invariant solutions depend on the specification of the function p.

Submodel 2.1. The operator $\langle X_1, X_3 \rangle$ has invariants

$$J_1 = t, \quad J_2 = \rho_1, \quad J_3 = \rho_2, \quad J_4 = u_1 - u_2, \quad J_5 = E_0.$$

Integer subgroup characteristics have the following values: $r_*(\xi) = 1$, $r_* = 2$, $t_* = 5$, $\sigma_* = 1$, $\mu_* = 4$. Let us find the corresponding partially invariant solutions. As the rank of a partially invariant solution, one can take any integer that ρ_* that satisfies the inequalities $\sigma_* \le \rho_* < \min(n, t_*)$, in this case $\rho_* = \sigma_* = 1$.

Thus, a regular partially invariant solution is sought in the form

$$\rho_1 = \rho_1(t), \quad \rho_2 = \rho_2(t), \quad u_1 = u_2 + \varphi(t), \quad E_0 = E_0(t).$$

The defect δ of this solution is equal to 1. Substituting the presented functions into the system of equations (2.1)–(2.4), we obtain

$$\frac{\partial \rho_1}{\partial t} + \rho_1 \frac{\partial u_2}{\partial x} = 0, \quad \frac{\partial \rho_2}{\partial t} + \rho_2 \frac{\partial u_2}{\partial x} = 0, \tag{2.14}$$
$$\frac{\partial u_2}{\partial t} + \frac{\partial \varphi}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + \varphi \frac{\partial u_2}{\partial x} + b\frac{\rho_2}{\rho_1}\varphi = 0, \tag{2.15}$$
$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} - b\varphi = 0, \tag{2.16}$$
$$\frac{\partial E_0}{\partial t} + (p + E_0 + \rho_1 \varphi^2)\frac{\partial u_2}{\partial x} + b\rho_1\varphi^2 = 0.$$

Subtracting the first equation from (2.14) multiplied by ρ_1 from the second one multiplied by ρ_2, we receive

$$\frac{\partial}{\partial t}\Big(\frac{\rho_2}{\rho_1}\Big) = 0, \quad \text{hence } \rho_2 = c_1 \rho_1.$$

Therefore,

$$u_{2x} = -\frac{\rho_1'}{\rho_1}, \quad u_2 = -\frac{\rho_1'}{\rho_1}x + c_2(t),$$

where prime denotes the derivative with respect to t.

Using the definition of the partial density for two-velocity medium components $\rho_i = \alpha_i \rho_i^{\text{ph}}$ (where α_i and ρ_i^{ph} are the volume concentrations and physical densities of the components), after simple transformations we arrive at the equation

$$\Big(1 - \frac{c_1\rho_1(t)}{\rho_2^{\text{ph}}}\Big)\Big(\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x}\Big) - \frac{\rho_1(t)}{\rho_2^{\text{ph}}}\Big(\frac{\partial}{\partial t}(u_2 + \varphi(t)) + (u_2 + \varphi(t))\frac{\partial u_2}{\partial x}\Big) = b\varphi(t). \tag{2.17}$$

Differentiate this expression with respect to x and take into account that $u_{2xx} = 0$:

$$\Big(1 - \frac{(c_1+1)\rho_1(t)}{\rho_2^{\text{ph}}}\Big)\Big(\frac{\partial u_{2t}}{\partial x} + \Big(\frac{\partial u_2}{\partial x}\Big)^2\Big) = 0.$$

Two cases are possible.

In the first case we have $\rho_1(t) = \frac{\rho_2^{\text{ph}}}{c_1+1}$ and $\rho_2(t) = \frac{c_1\rho_2^{\text{ph}}}{c_1+1}$. Provided that $\rho_2^{\text{ph}} = \text{const}$, we obtain $u_2 = u_2(t)$. Then the equations (2.15) and (2.16) take the form

$$\frac{\partial}{\partial t}\varphi(t) + b(c_1+1)\varphi(t) = 0, \quad \frac{\partial}{\partial t}u_2 = b\varphi(t).$$

Hence we have $\varphi(t) = c_3\exp(-b(c_1+1)t)$. Thus, we obtain the solution

$$\rho_1 = \frac{\rho_2^{\text{ph}}}{c_1+1}, \quad \rho_2 = \frac{c_1\rho_2^{\text{ph}}}{c_1+1},$$

$$u_1 = \frac{c_1c_3}{c_1+1}\exp(-b(c_1+1)t), \quad u_2 = -\frac{c_3}{c_1+1}\exp(-b(c_1+1)t),$$

$$E_0 = \frac{\rho_2^{\text{ph}}}{2}\Big(\frac{c_3}{c_1+1}\Big)^2\exp(-b(c_1+1)t) + c_4.$$

In the second case $\partial_{tx}u_2 + \big(\frac{\partial}{\partial x}u_2\big)^2 = 0$. Let us denote $\psi(t) = \frac{\partial}{\partial x}u_2 = -\frac{\rho_1'(t)}{\rho_1(t)}$, then we have the equation $\frac{\partial}{\partial t}\psi(t) + \psi^2(t) = 0$. Its solution is $\psi(t) = \frac{1}{t+c_5}$, what entails $-\frac{\rho_1'(t)}{\rho_1(t)} = \frac{1}{t+c_5}$ and $\rho_1(t) = \frac{c_6}{t+c_5}$, $u_2 = \frac{x}{t+c_5} + c_2(t)$. From (2.16) it follows that $\varphi(t) = \frac{1}{b}\big(\frac{\partial}{\partial t}c_2(t) + \frac{c_2(t)}{t+c_5}\big)$. From (2.15):

$$\frac{\partial}{\partial t}\varphi(t) + \Big(b(1+c_1) + \frac{1}{t+c_5}\Big)\varphi(t) = 0.$$

Then

$$\varphi(t) = \frac{c_6}{t + c_5} \exp(-b(c_1 + 1)t), \quad \frac{\partial}{\partial t} c_2(t) + \frac{c_2(t)}{t + c_5} = \frac{bc_6}{t + c_5} \exp(-b(c_1 + 1)t)$$

and

$$c_2(t) = \frac{c_7}{t + c_5} - \frac{c_6}{(c_1 + 1)(t + c_5)} \exp(-b(c_1 + 1)t).$$

Thus, we obtain the solution

$$\rho_1(t) = \frac{c_6}{t + c_5}, \quad \rho_2(t) = \frac{c_1 c_6}{t + c_5},$$

$$u_1 = \frac{x}{t + c_5} + \frac{c_1 c_6}{(c_1 + 1)(t + c_5)} \exp(-b(c_1 + 1)t) + \frac{c_7}{t + c_5},$$

$$u_2 = \frac{x}{t + c_5} - \frac{c_6}{(c_1 + 1)(t + c_5)} \exp(-b(c_1 + 1)t) + \frac{c_7}{t + c_5},$$

$$\frac{\partial}{\partial t} E_0 + (p + E_0(t) + \rho_1(t)\varphi^2(t)) \frac{1}{t + c_5} + b\rho_1(t)\varphi^2(t) = 0.$$

The invariant solutions depend on the specification of the function p.

Submodel 2.2a. Operator $\langle X_1, X_2 \rangle$ has invariants

$$J_1 = \rho_1, \quad J_2 = \rho_2, \quad J_3 = u_1, \quad J_4 = u_2, \quad J_5 = E_0.$$

The solution is sought in the form

$$\rho_1 = c_1, \quad \rho_2 = c_2, \quad u_1 = c_3, \quad u_2 = c_4, \quad E_0 = c_5.$$

Substituting these solutions into the system gives the solution

$$\rho_1 = c_1, \quad \rho_2 = c_2, \quad u_1 = u_2 = c_3, \quad E_0 = c_5.$$

Submodel 2.2b. The operator $\langle X_1, X_2 + cX_3 \rangle$ has invariants

$$J_1 = \rho_1, \quad J_2 = \rho_2, \quad J_3 = u_1 - ct, \quad J_4 = u_2 - ct, \quad J_5 = E_0.$$

The solution is sought in the form

$$\rho_1 = c_1, \quad \rho_2 = c_2, \quad u_1 = c_3 + ct, \quad u_2 = c_4 + ct, \quad E_0 = c_5.$$

Substituting these solutions into equations (2.1) will not give anything. Substituting the required functions into equations (2.2)–(2.4) gives:

$$c = -b\frac{c_2}{c_1}(c_3 - c_4), \quad c = b(c_3 - c_4), \quad bc_1(c_3 - c_4) = 0.$$

The last relation implies $c_3 = c_4$, and hence $c = 0$. Thus, there are no invariant solutions for this operator.

Submodel 3.1. For the operator $\langle X_1, X_2, X_3\rangle$, the invariants are the functions

$$J_1 = \rho_1, \quad J_2 = \rho_2, \quad J_3 = u_1 - u_2, \quad J_4 = E_0.$$

In this case, there are no invariant solutions. Meanwhile, for the subalgebra $\langle X_1, X_2, X_3\rangle$ we have the following values $r_*(\xi) = 2$, $r_* = 3$, $t_* = 4$, $\sigma_* = 0$, $\mu_* = 4$. For these values, it is possible to consider the case of existence the irregular partially invariant solution of the simple-wave type. Let us take the function u_1 as a parameter. Then we have a representation for invariant functions

$$J_1 = \rho_1(u_1), \quad J_2 = \rho_2(u_1), \quad J_3 = u_2(u_1), \quad J_4 = E_0(u_1).$$

In this case, the non-invariant function u_1 is assumed to depend on all independent variables $u_1 = u_1(t, x)$. The defect δ of this solution is equal to 1. Substituting the desired functions into the system of equations (2.1)–(2.4), we obtain

$$\begin{aligned}
&\rho_1' u_{1t} + (\rho_1 + u_1\rho_1')u_{1x} = 0, \quad \rho_2' u_{1t} + (\rho_2 u_2' + u_2\rho_2')u_{1x} = 0,\\
&u_{1t} + u_1u_{1x} = -\frac{1}{\rho}\big(p_\rho(\rho_1' + \rho_2') + 2p_\omega w(1 - u_2') + p_{E_0}E_0'\big)u_{1x} -\\
&\qquad \frac{\rho_2 w}{\rho}(1 - u_2')u_{1x} - b\frac{\rho_2}{\rho_1}w,\\
&u_2' u_{1t} + u_2u_2' u_{1x} = -\frac{1}{\rho}\big(p_\rho(\rho_1' + \rho_2') + 2p_\omega w(1 - u_2') + p_{E_0}E_0'\big)u_{1x} +\\
&\qquad \frac{\rho_1 w}{\rho}(1 - u_2')u_{1x} + bw,\\
&E_0' u_{1t} + \rho^{-1} jE_0' u_{1x} + b\rho_1\omega + \rho^{-2}(\rho_2 w(p + E_0) - \rho_2^2 w^3)\rho_1' u_{1x} -\\
&\qquad \rho^{-2}(\rho_1 w(p + E_0) + \rho_1^2 w^3)\rho_2' u_{1x} +\\
&\qquad \rho^{-1}(\rho_1(p + E_0) + \rho_1^2\omega + 3\rho_1\rho_2\omega)u_{1x} +\\
&\qquad \rho^{-1}(\rho_2(p + E_0) - 2\rho_1\rho_2\omega)u_2' u_{1x} = 0,
\end{aligned}$$

where the prime denotes the derivative with respect to u_1. Partially invariant solutions depend on the specification of the function p.

3. Numerical solution of the dissipative system of equations for a two-phase medium in the two-dimensional case

The mathematical model of two-velocity medium motion in a gravitational field (for isentropic case) has the form

$$\frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1 \boldsymbol{u}_1) = 0, \quad \frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_2 \boldsymbol{u}_2) = 0, \tag{3.1}$$

$$\frac{\partial}{\partial t}(\rho_1 u_{1,i} + \rho_2 u_{2,i}) + \frac{\partial}{\partial x_j}(\rho_1 u_{1,i} u_{1,j} + \rho_2 u_{2,i} u_{2,j} + p\delta_{ij}) = \rho g_i, \tag{3.2}$$

$$\frac{\partial}{\partial t}(u_{1,i} - u_{2,i}) + u_{1,j}\frac{\partial u_{1,i}}{\partial x_j} - u_{2,j}\frac{\partial u_{2,i}}{\partial x_j} + \frac{1}{2}\frac{\partial}{\partial x_i}(u_{1,j} - u_{2,j})^2 + b\frac{\rho}{\rho_1}(u_{1,i} - u_{2,i}) = 0, \tag{3.3}$$

where g_i is the gravitational acceleration, $b = \eta/\rho k$ is the friction coefficient, η is the viscosity of the dispersed phase, and k is the permeability.

The numerical implementation of the system of equations (3.1)–(3.3) in the two-dimensional case is based on the following computational algorithm. We consider a finite-volume discretization of the equation system (3.1)–(3.3) for a spatial cell making a uniform grid in a rectangular coordinate system:

$$\frac{dU_{i,j}}{dt} + \frac{F_{i+1/2,j} - F_{i-1/2,j}}{\Delta x} + \frac{G_{i,j+1/2} - G_{i,j-1/2}}{\Delta y} = \mathrm{Q}(x_i, y_j, U_{i,j}). \tag{3.4}$$

Solution for (3.4) reduces to integrating a system of ordinary differential equations if the values of the fluxes and at the boundaries between cells are known. For the numerical integration of this equation, the Runge–Kutta method of the third order is used [11]

$$\begin{aligned}
U^{(0)} &= U^n, \\
U^{(i)} &= U^{(i-1)} + \frac{1}{2}\Delta t L(U^{(i-1)}), \quad i = 1, \dots, m-1, \\
U^{(m)} &= \sum_{k=0}^{m-2} \alpha_{m,k} U^{(k)} + \alpha_{m,m-1}\Big(U^{(m-1)} + \frac{1}{2}\Delta t L(U^{(m-1)})\Big), \\
U^{n+1} &= U^{(m)},
\end{aligned}$$

where

$$L(U) = -\frac{F_{i+1/2,j} - F_{i-1/2,j}}{\Delta x} - \frac{G_{i,j+1/2} - G_{i,j-1/2}}{\Delta y} + \mathrm{Q}(x_i, y_j, U_{i,j}),$$

$m = 4$, and the coefficients $\alpha_{m,k}$ are chosen as follows: $\alpha_{4,0} = 0$, $\alpha_{4,1} = 2/3$, $\alpha_{4,2} = 0$, $\alpha_{4,3} = 1/3$. Here U^n and U^{n+1} are the values of the solution at the nth and $(n+1)$th time layers.

The third-order Runge–Kutta method used in numerical integration belongs to the class of temporal SSP discretization, which makes it possible to maintain monotonicity and increase the order of approximation of circuits in time. In this case, the spatial derivatives contained in the vector of the right parts $\mathrm{Q}(x_i, y_j, U_{i,j})$ are approximated by operators of the central finite difference. The flux values at the cell boundaries are calculated by the GFORCE method [12] with the fluxes calculated as a combination of Lax–Friedrichs and two-step Lax–Wendroff fluxes:

$$F^{\mathrm{GF}}_{i+1/2} = \theta F^{\mathrm{LW}}_{i+1/2} + (1-\theta) F^{\mathrm{LF}}_{i+1/2}, \quad 0 \le \theta \le 1,$$

$$F^{\mathrm{LF}}_{i+1/2} = \frac{1}{2}(F(U^n_i) + F(U^n_{i+1})) - \frac{1}{2}\frac{\Delta x}{\Delta t}(U^n_{i+1} - U^n_i), \tag{3.5}$$

$$F^{\mathrm{LW}}_{i+1/2} = F(U^{\mathrm{LW}}_{i+1/2}), \tag{3.6}$$

$$U^{\mathrm{LW}}_{i+1/2} = \frac{1}{2}(U^n_i + U^n_{i+1}) - \frac{1}{2}\frac{\Delta t}{\Delta x}(F(U^n_{i+1}) - F(U^n_i)).$$

The time step is calculated according to the formula

$$\Delta t = K_{\mathrm{CFL}} \min_i \frac{\Delta x}{C^{\max}_i},$$

where K_{CFL} is the Courant number and $C^{\max}_i$ is the maximum speed of sound.

We will use the GFORCE method in conjunction with TVD reconstruction with a minmod type limiter [13]. Then flows (3.5) and (3.6) will be calculated as follows

$$F^{\mathrm{LF}}_{i+1/2} = \frac{1}{2}(F(U^n_L) + F(U^n_R)) - \frac{1}{2}\frac{\Delta x}{\Delta t}(U^n_R - U^n_L),$$

$$F^{\mathrm{LW}}_{i+1/2} = F(U^{\mathrm{LW}}_{i+1/2}), \quad U^{\mathrm{LW}}_{i+1/2} = \frac{1}{2}(U^n_L + U^n_R) - \frac{1}{2}\frac{\Delta t}{\Delta x}(F(U^n_R) - F(U^n_L)),$$

where

$$U_L = U^n_i - \frac{1}{2}\sigma_i, \quad U_R = U^n_R + \frac{1}{2}\sigma_i,$$

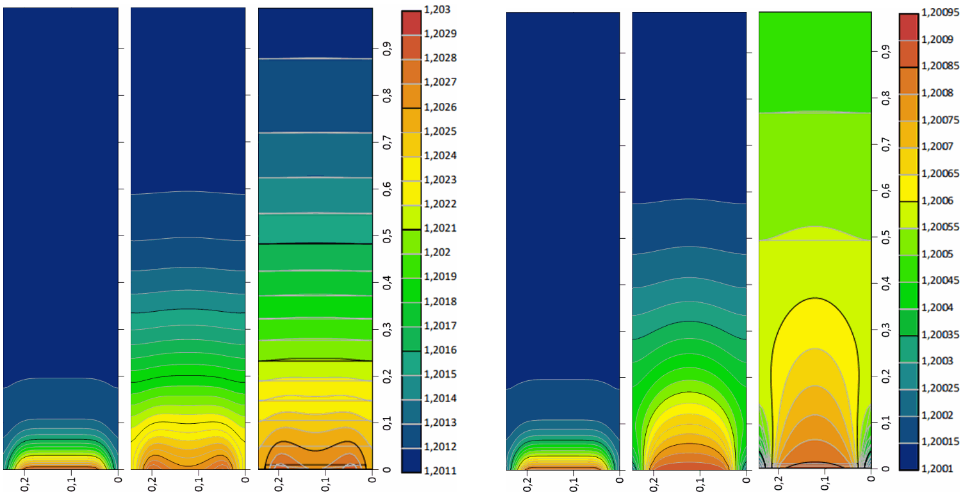
$$\sigma_i = \begin{cases} \max(0, \min(\Delta_{i-1/2}, \Delta_{i+1/2})), & \Delta_{i+1/2} \ge 0, \\ \min(0, \max(\Delta_{i-1/2}, \Delta_{i+1/2})), & \Delta_{i+1/2} < 0, \end{cases}$$

$$\Delta_{i-1/2} = U^n_i - U^n_{i-1}, \quad \Delta_{i+1/2} = U^n_{i+1} - U^n_i.$$

The TVD scheme is based on the principle of non-increase in the total variation of the solution, where a higher order of the numerical scheme is achieved using the polynomial reconstructions of quantities and the use of limiter functions (limiters). The TVD scheme is a high-precision method for hydrodynamic problems with discontinuous solutions, which is a nonlinear scheme that can ensure the solution monotonicity. At the boundary points of the enumeration domain, one-dimensional problems for boundary discontinuity decay are solved. The systems of equations describing the decay of the discontinuity along the corresponding coordinate axes are obtained as a result of linearization of the model equations and neglecting the dissipative terms (they violate self-similarity of solution).

The computational algorithm is implemented as a set of C++ programs using the Microsoft Visual Studio integrated software development environment. In addition to standard libraries, the Armadillo linear algebra library [14, 15] is used: the functions of the library are calculating the eigenvalues and corresponding eigenvectors of a dense general (non-symmetric, non-Hermitian) square matrix. In order to minimize the machine rounding errors, numerical solution calculations are performed in dimensionless variables.

As a model system, we considered the motion of a heterophase melt in a magma channel during its intrusion into a permeable zone of the crust. A non-uniform distribution of the solid phase is set at the inlet boundary. On the left boundary, the longitudinal components of the phase velocities are set (equal or different). The right border is considered open. On the lateral boundaries, either no-slip conditions or slip conditions are assigned. The phase density distributions in a vertically oriented channel are shown in Figures 1, 2. The distribution of the total density and the content of dispersed particles in an inclined channel are shown in Figures 3 and 4. The phase density distributions in the inclined channel are shown in Figure 5.



a b

Figure 1. Evolution of the solid phase density for a heterophase medium in a vertical channel with gravity field under the condition of (a) slipping, (b) sticking at the side boundaries at different times

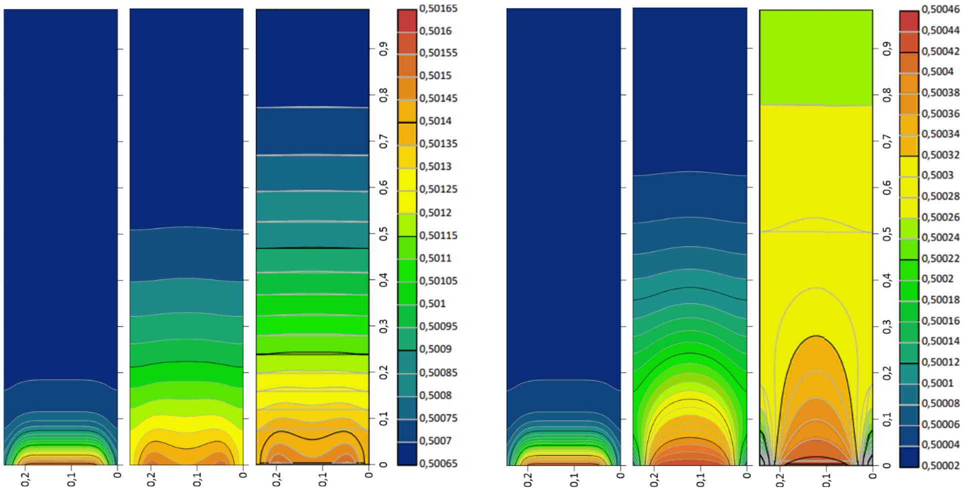


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Figure 2. Evolution of the liquid phase density for a heterophase medium in a vertical channel with gravity under the condition of (a) slipping, (b) sticking at the side boundaries at different times

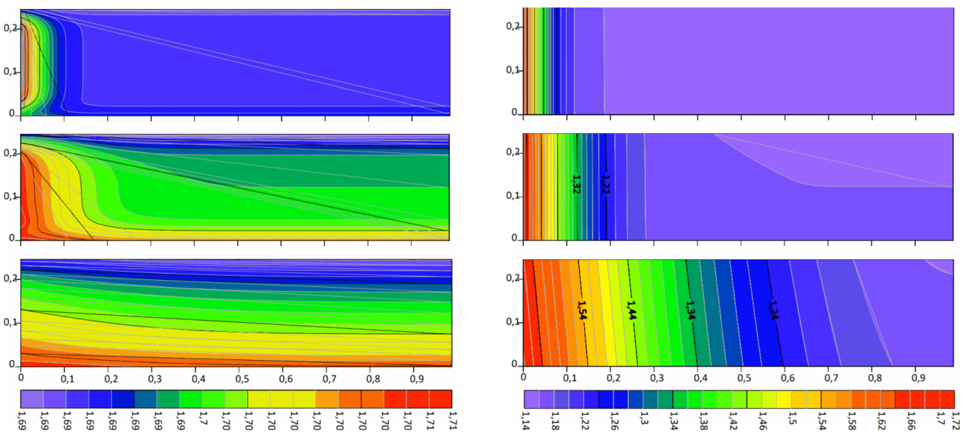

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Figure 3. Density evolution during the flow of a heterophase medium in an inclined channel in a gravitational field under the condition of (a) slip, (b) sticking at the lateral boundaries at different times

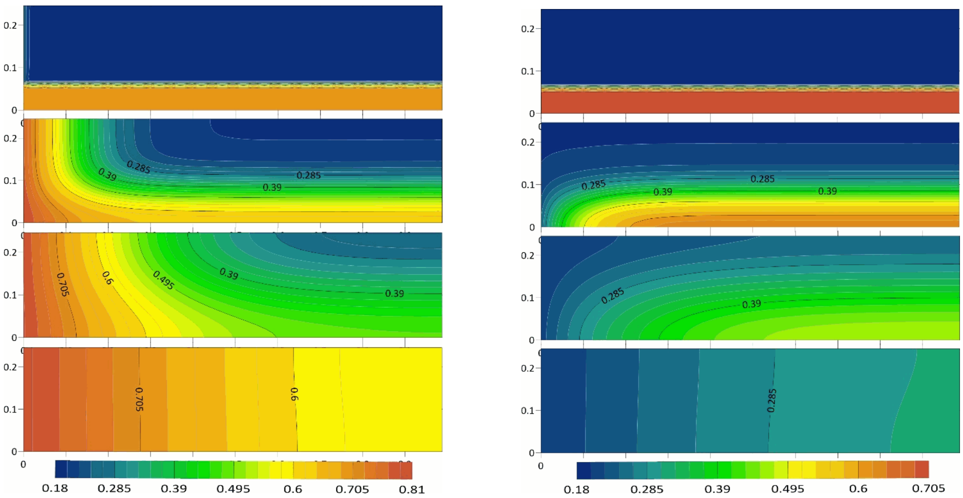

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Figure 4. Evolution of the solids content in the flow during intrusion of a heterophase medium with (a) high content, (b) low content of the solid phase into an inclined channel with an inhomogeneous initial distribution of solids

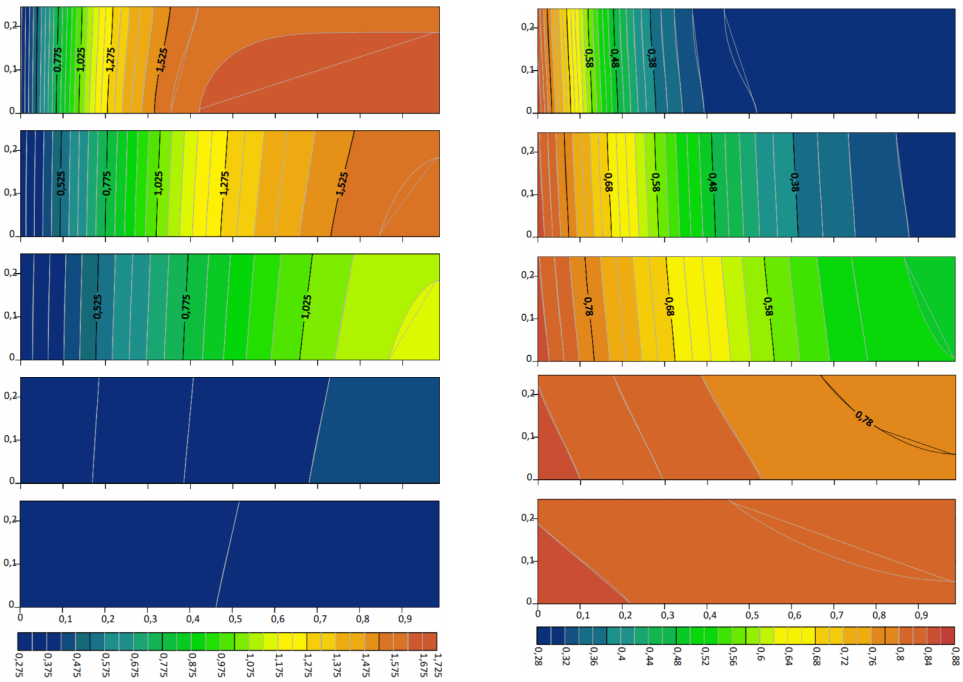

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Figure 5. Evolution of the densities of (a) solid phase, (b) liquid phase during the flow of a heterophase medium in an inclined channel in a gravitational field at different times

Conclusion

The paper considers a dissipative system of equations for the dynamics of a two-phase medium. The study of symmetries was carried out for a one-dimensional system of equations where dissipation due to interfacial friction was considered. The core basis of the main Lie algebra, the optimal systems of subalgebras are found, and all invariant and partially invariant submodels are written out. A trivial invariant solution, invariant solutions of three submodels of rank 1, a regular partially invariant solution of a submodel of rank 2, an irregular partially invariant solution of the simple wave type of a submodel of rank 3 are demonstrated. The flux values at the cell boundaries are calculated by the GFORCE method, where the fluxes are calculated as a combination of Lax–Friedrichs and two-step Lax–Wendroff fluxes. The problem of two-phase medium motion in a channel with gravity is solved for various boundary conditions with an inhomogeneous initial distribution of the dispersed phase.

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