Bull. Nov. Comp. Center, Num. Anal., 9 (2000), 105–110

Properties of matrices in methods of constructing an interpolating spline via the coordinates of its derivatives in B-spline basis*

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We consider an interpolation problem. We have the given values $f_i = f(x_i)$ of some periodic function $f(x)$ of the period $b-a$ in the nodes of a mesh

$$\Delta: \quad a = x_0 < x_1 < \ldots < x_N = b.$$

It is required to construct a $(b-a)$-periodic interpolating spline $S(x)$ of the degree $2n-1$ ($N \geq 2n$).

In practice, the construction of a spline (of the minimal defect) is reduced to solution of some system of linear equations. For splines of arbitrary odd degree the method of determining the B-spline decomposition coefficients is most widespread. However it is not optimal. For example, in the case of cubic splines the priority is usually given to methods of construction via the first or second derivatives of a spline in the nodes, as for any nonuniform mesh the condition number of matrices of the corresponding systems is equal to 3. But the conditioning of the B-spline collocation matrix on essentially nonuniform meshes can be arbitrarily large [1]. Similar estimates of conditioning were obtained by the author for splines of arbitrary odd degree [2]. But in general case, contrary to the cubic one, methods with a well conditioned system of equations are unknown. Generally, to derive systems with respect to the nodal values of some derivative is a rather complicated problem. Only the system with respect to the moments (with respect to the $(2n-2)$-th derivative, if the spline degree is equal to $2n-1$) is known [3]. But the matrix of this system on a nonuniform mesh can also be ill conditioned [4].

The author has derived the systems of equations for finding the derivatives of splines as coordinates with respect to a B-spline basis [5]. In the present paper, the properties of matrices of such systems are considered.

The condition number of a matrix $\boldsymbol{A}$ is understood as the value

$$\operatorname{cond}(\boldsymbol{A}) = \|\boldsymbol{A}\| \cdot \|\boldsymbol{A}^{-1}\|$$

*Supported by SB RAS under Grant 1–2000.

with the Chebyshev max-norm of a matrix. We assume that the mesh Δ is periodically continued beyond the interval $[a,b]$ over the whole of the real line. Let $\bar{h}$ denote the length of the largest interval of Δ. The B-splines which we use have the numbering which is different from the standard one (cf. [1, 6, 7]) but is more convenient for our purposes, – the index of a B-spline corresponds to the number of the middle knot on the support, but not the left one, i.e., B-splines of the degree $k-1$ (of order k) $B_{ik}(x)$ and $Q_{ik}(x)$ have the support $(x_{i-[k/2]}, x_{i+[(k+1)/2]})$. The above B-splines differ by the normalization

$$\sum_i B_{ik}(x) \equiv 1, \qquad \int\limits_{\operatorname{supp} Q_{ik}} Q_{ik}(\tau)\,d\tau = 1,$$

$$Q_{ik}(x) = \frac{k}{x_{i+[(k+1)/2]} - x_{i-[k/2]}} B_{ik}(x).$$

The bases formed by corresponding periodic B-splines (see [7]) will be denoted by $\widetilde{B}_{1k}, \widetilde{B}_{2k}, \dots, \widetilde{B}_{Nk}$ and $\widetilde{Q}_{1k}, \widetilde{Q}_{2k}, \dots, \widetilde{Q}_{Nk}$ respectively.

For $1 \le k \le 2n-1$, the author has derived ([5]) the systems of equations

$$\boldsymbol{A}_k \boldsymbol{b}_k = \boldsymbol{f}^{(k)} \tag{1}$$

with respect to $b_{1k}, b_{2k}, \dots, b_{Nk}$ – the coefficients of the decomposition of the k-th derivative of the required spline $S(x)$ in normalized periodic B-splines of the degree $2n-1-k$, i.e.,

$$S^{(k)}(x) = \sum_{j=1}^{N} b_{jk} \widetilde{B}_{j,2n-k}(x). \tag{2}$$

The elements a_{ij}^k of the matrices $\boldsymbol{A}_k$ are of the form

$$a_{ij}^k = \int\limits_a^b \widetilde{Q}_{ik}(\tau) \widetilde{B}_{j,2n-k}(\tau)\,d\tau,$$

and the elements $f_i^{(k)}$ of the vector $\boldsymbol{f}^{(k)}$ on the right-hand side of (1) are equal to

$$f_i^{(k)} = k! \cdot f\Big[x_{i-[k/2]}, \dots, x_{i+[(k+1)/2]}\Big].$$

The properties of B-splines directly imply that all matrices $\boldsymbol{A}_k$ are $(2n-1)$-banded, and their elements a_{ij}^k have the properties

$$a_{ij}^k \ge 0, \qquad i,j = 1,2,\dots,N;$$

$$\sum_{j=1}^{N} a_{ij}^k = 1, \qquad i = 1,2,\dots,N.$$

It means that $\|A_k\| = 1$ and, hence, $\text{cond}(A_k) = \|A_k^{-1}\|$.

Lemma 1. *In the special case when Δ is uniform, the matrices A_k of all systems* (1) *for $k = 1, 2, \ldots, 2n-1$ are identical and coincide with the B-spline collocation matrix, i.e.,*

$$a_{ij}^k = B_{i,2n}(x_j).$$

Proof. Let h be the step of a uniform mesh. Without any loss of generality one can take $x_i = ih$. The formula for the convolution of B-splines [7] in our notations is

$$\int_0^{x_k} B_{[\frac{k}{2}],k}(x-y) B_{[\frac{2n-k}{2}],2n-k}(y)\, dy = B_{n,2n}(x). \tag{3}$$

Assume $x = nh + ih - jh$. Then the expression on the right-hand side of (3) is equal to $B_{i,2n}(x_j)$.

Since $B_{[k/2],k}(x-y) = B_{[k/2],k}(kh - x + y)$ and $B_{lm}(\tau) = B_{l+1,m}(\tau + h)$, therefore at $x = nh + ih - jh$ we have

$$B_{[\frac{k}{2}],k}(x-y) = B_{ik}(y+\sigma), \quad B_{[\frac{2n-k}{2}],2n-k}(y) = B_{j,2n-k}(y+\sigma)$$

with $\sigma = jh - \left[\frac{2n-k}{2}\right] h$. So, the left-hand side of (3) is equal to

$$\int_{\text{supp}(B_{ik}B_{j,2n-k})} B_{ik}(\tau) B_{j,2n-k}(\tau)\, d\tau.$$

Since in this case the B-splines B_{ik} and Q_{ik} coincide, the left-hand side of (3) is equal to a_{ij}^k. □

Lemma 2 [5]. *For $f \in C^k[a,b]$ the following error bound holds*

$$\|S^{(k)} - f^{(k)}\|_\infty \le \left(n - \left[\frac{k}{2}\right] + (n-1+k)\|A_k^{-1}\|\right) \omega(f^{(k)}; \bar{h}). \tag{4}$$

As it is known, the influence of rounding errors when solving a system of equations is characterized by the condition number of the matrix of the system. Thus, the good conditioning of A_k not only guarantees good accuracy of calculating the spline $S(x)$ in the result of solving (1), but also reduces the constant in (4). However, good approximation of $f^{(k)}(x)$ by the spline $S^{(k)}(x)$ does not imply good conditioning of the matrix A_k.

Let us consider a sequence of meshes $\Delta_\nu : a = x_{0\nu} < x_{1\nu} < \ldots < x_{N\nu} = b$, $\nu = 1, 2, \ldots$, such that

$$\bar{h}_\nu = \max_i (x_{i+1,\nu} - x_{i\nu}) \to 0 \quad \text{as } \nu \to \infty. \tag{5}$$

Let a $(b-a)$-periodic spline $S_\nu(x)$ of the degree $2n-1$ interpolates the function $f(x)$ at the nodes of the mesh Δ_ν.

Theorem 1. *If the matrix $A_{k\nu}^{-1}$, $k = 1, 2, \dots, 2n-1$, which is inverse to the matrix $A_{k\nu}$ of the coefficients of the system of equations* (1) *is uniformly bounded with respect to ν, then*

$$S_\nu^{(k)}(x) - f^{(k)}(x) = o(1) \tag{6}$$

uniformly with respect to x on $[a, b]$.

The proof immediately follows from Lemma 2.

It was shown by the author [4] that if $k = 1, 2, \dots, n-2$ or $k = n+1, n+2, \dots, 2n-1$, then there exists a function $f(x) \in C^k[a, b]$ and a sequence of meshes $\{\Delta_\nu\}$ possessing property (5) such that

$$\left\| S_{\Delta_\nu}^{(k)} - f^{(k)} \right\|_\infty \to \infty \quad \text{as } \nu \to \infty, \tag{7}$$

i.e., relation (6) does not hold. Thus, the following theorem takes place.

Theorem 2. *If $k = 1, 2, \dots, n-2$ or $k = n+1, n+2, \dots, 2n-1$, then for any arbitrarily large constant K there exists a mesh Δ such that* $\operatorname{cond}(A_k) = \|A_k^{-1}\| > K$.

It should be noted that though $\|A_k^{-1}\|$, $k = 1, 2, \dots, n-2$, can be arbitrarily large, nevertheless, for a rather smooth function $f(x)$ (for example, $f(x) \in C^n[a, b]$) relation (6) holds [3].

Theorem 2 leaves only two systems of equations ($k = n-1, n$) which, probably, are well conditioned for arbitrary nonuniform meshes Δ. Concerning the boundedness of the value $\|A_n^{-1}\|$, the assumption was made as early as in 1973. It is checked for $n = 2, 3$ [8]. For an arbitrary n only estimates depending either on the global mesh ratio [9] or on the number of nodes of a mesh [10] were established.

Numerous computing experiments which we have conducted show that the worst from the viewpoint of conditioning of systems (1) are the geometrical meshes which were used for the construction of an example of divergence of the interpolating process [4], i.e.,

$$\Delta: \ -1 = x_0 < x_1 < \dots < x_N = 1;$$

$$x_{i+1} = x_i + \rho(x_i - x_{i-1}), \qquad i = 0, 1, \dots, [N/2] - 1;$$
$$x_{N-i} = x_i, \qquad i = 0, 1, \dots, [N/2].$$

Tables 1 and 2 contain the values $\operatorname{cond}(A_{n-1})$ and $\operatorname{cond}(A_n)$ respectively for the splines of odd degrees from 3 up to 15 calculated on geometrical (with

Table 1. The condition numbers of the matrices A_{n-1}

ρ	N	Degree of $S(x)$						
		3	5	7	9	11	13	15
1	20	3.0	7.5	18.53	45.73	112.8	278.4	686.9
	50	3.0	7.5	18.53	45.73	112.8	278.4	686.9
	100	3.0	7.5	18.53	45.73	112.8	278.4	686.9
5	20	4.83	19.37	58.47	202.6	616.2	2252.	7587.
	50	4.83	19.52	60.32	213.9	658.4	2394.	7965.
	100	4.83	19.52	60.34	214.1	660.5	2406.	8013.
15	20	5.54	22.81	65.59	227.2	685.2	2534.	8531.
	50	5.54	23.08	68.14	240.5	729.7	2673.	8883.
	100	5.54	23.08	68.17	240.9	732.5	2687.	8933.

Table 2. The condition numbers of the matrices A_n

ρ	Degree of $S(x)$						
	3	5	7	9	11	13	15
1	3.0	7.5	18.53	45.73	112.8	278.4	686.9
5	3.0	9.22	34.38	117.7	472.2	1702.	6921.
15	3.0	9.72	39.26	135.5	552.6	1973.	7995.

a parameter ρ) meshes for various N and ρ. The condition numbers of the matrices A_n for various ρ weakly react to changes of N. Since for each ρ the calculated values of the condition numbers for $N = 20, 50, 100$ coincide to within all digits given in the table, therefore the column (N), containing the number of nodes in the mesh, is omitted in Table 2.

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