Bull. Nov. Comp. Center, Num. Anal., 10 (2001), 61–66

Condition number of collocation method on a quasiuniform grid for the integral equation of the 1st kind with logarithmic singularity*

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The paper is devoted to the substantiation of the direct numerical method for the integral equation of the first kind with logarithmic singularity on the closed curve; it is based on the piecewise-linear approximation of unknown function from its values at quasiuniform grid and collocation condition in the same gridpoints. As it is shown, if the initial integral equation is uniquely solvable and the quasiuniform parameter does not exceed 23.3, then for a step small enough the condition number of a matrix obtained in this method is not less than a constant multiplied by the step.

We consider the equation of the 1st kind defined initially on a smooth closed curve. If we parameterize the curve and separate the logarithmic singularity from the kernel, then the equation may be written in the form:

$$(\mathcal{K}f)(t) \equiv \int_{-\pi}^{\pi} \left[\log \frac{A}{2\sin\frac{\tau-t}{2}} + K_1(t,\tau)\right] f(\tau)\, d\tau = g(t), \quad t \in [-\pi, \pi], \quad (1)$$

where A is a constant not equal to 1. All functions are 2π-periodic.

Let us call the equation of this type "almost-model", if the junior term of its kernel $K_1(t,\tau)$ is smooth enough (as a rule, infinitely smooth). Then there are some simple operations, reducing the equation to a singular integral equation with a kernel of the Hilbert type or to the Fredholm equation of 2nd kind with a smooth junior term. These operations are the differentiation of both sides or multiplying the equation by the operator which is inverse to the main part of the integral operator. After that, we can solve the new equation numerically using many well-known methods.

But if equation (1) is an equation of general form, then the junior term $K_1(t,\tau)$ may contain singularities of the form $r \log r$, $r^2 \log r$ and so on. In this case, the simple analytic operations mentioned above do not make the problem easier because the real accuracy of numerical method will mainly depend on the accuracy of approximation of a junior term. So fulfilling the differentiation or applying the inverse operator, we do not gain anything,

*Supported by the Russian Foundation for Basic Research under Grant 00-01-00901.

therefore these operations are useless. Furthermore, if the kernel is given in any complicated form, the fulfilling the analytic operations may cause some difficulties.

Such troubles do often arise in diffraction problems, (for example, see [1]); the kernels of integral operators do there consist of any combinations of functions similar to the Hankel ones. Therefore an immediate discretization of the initial equation (1) can be the most suitable method for such problems.

Applying the interpolation and collocation method, let us define any gridpoints $\{\tau_j;\ j = -N_1, \dots, N_2\}$ on the segment $[-\pi, \pi]$ and interpolate the unknown function $f(\tau)$ piecewise-linearly from its values at these gridpoints, i.e., represent f in the form

$$f(\tau) = \sum_{j=-N_1}^{N_2} a_j\, \varphi_j(\tau), \tag{2}$$

where φ_j are the basic functions of this interpolation:

$$\varphi_j(\tau) = \begin{cases} (\tau - \tau_{j-1})/(\tau_j - \tau_{j-1}), & \text{if} \quad \tau \in [\tau_{j-1}, \tau_j] \\ (\tau_{j+1} - \tau)/(\tau_{j+1} - \tau_j), & \text{if} \quad \tau \in [\tau_j, \tau_{j+1}] \\ 0 & \text{elsewhere.} \end{cases} \tag{3}$$

Let us search the unknown coefficients a_j from the collocation conditions at the same gridpoints:

$$\mathcal{K} f(\tau_k) \equiv \sum_{j=-N_1}^{N_2} a_j (\mathcal{K}\varphi_j)(\tau_k) = g(\tau_k), \qquad k = -N_1, \dots, N_2. \tag{4}$$

If the initial integral operator is not degenerate, then the images $\mathcal{K}\varphi_j$ of basic functions are linearly independent. But it does not provide the solvability of system (4) with respect to $\{a_j\}$. What we really need is the independence of their restrictions onto the grid $\{\tau_j\}$. Moreover, since some estimates of stability of the method are desired, we should get an information about the condition number of the matrix of system (4).

In previous investigations of this problem (see [2–5]), the uniform grids were only considered; these results are based on the fact that trigonometric functions are eigenfunctions for the main part of the integral operator, and restrictions of images of their interpolants onto a uniform grid preserve the orthogonal properties. This approach fails on non-uniform grids.

The new approach uses convex properties of images of the basic functions, and so some quasiuniform grids may be considered.

Let us denote steps of the grid $h_{j+1/2} \equiv \tau_{j+1} - \tau_j$, let h be the minimum of them. Let us consider the sequence of grids such that the quantity

$$Q = \frac{\max_j h_{j+1/2}}{\min_j h_{j+1/2}}$$

(quasiuniform parameter) is the same for every grid.

Then the following theorem has been proved.

Theorem 1. *Let $f(\tau)$ be of form (2), $\max|a_j| = 1$, the quasiuniform parameter satisfies the condition $Q \le 23.3$, and $g = \mathcal{K}f$. Then, for sufficiently small h, there exists such a gridpoint τ_j that $|g(\tau_j)| \ge ch$, where c is any constant independent on h.*

Or, in other words, the condition number of the matrix of (4) is not less than ch.

The way of its proof consists of the considerations exposed below.

1. Firstly, the "standard" equation without the junior term is considered:

$$(\mathcal{K}_0 f)(t) \equiv \int_{-\pi}^{\pi} \log \frac{A}{2\sin\frac{\tau - t}{2}} f(\tau)\, d\tau = g(t), \quad t \in [-\pi, \pi]. \tag{5}$$

The following statement can be proved easily.

Theorem 2 (The case of non-zero average). *There exist positive constants c and C_* independent on h such that the following statement is valid: if $f(\tau)$ is a function of form (2), where $\max|a_j| = 1$, the average value of f has a modulus greater than $C_* h$, and h is small enough, then*

$$|g(\tau_j)| \equiv |\mathcal{K}_0 f(\tau_j)| > ch$$

at some gridpoint τ_j.

2. The further consideration uses the second finite differences. If i, j, k are three numbers of the gridpoints ($i < j < k$), then the second finite difference of a function g on these gridpoints is:

$$D_2(g; i, j, k) \equiv \frac{\tau_k - \tau_j}{\tau_k - \tau_i}[g(\tau_i) - g(\tau_j)] + \frac{\tau_j - \tau_i}{\tau_k - \tau_i}[g(\tau_k) - g(\tau_j)]$$

and the simplified finite difference is:

$$\tilde{D}_2(g; i, j, k) \equiv \frac{1}{2}[g(\tau_i) - g(\tau_j)] + \frac{1}{2}[g(\tau_k) - g(\tau_j)].$$

If

$$|D_2(g; i, j, k)| > ch \quad \text{or} \quad |\tilde{D}_2(g; i, j, k)| > ch, \tag{6}$$

where $g = \mathcal{K}_0 f$, then evidently at least one of the values $g(t)$ at these gridpoints has a modulus greater than ch. So it suffice to prove one of inequalities (6).

We have assumed before that $\max|a_j| = 1$, now let us suppose that namely $a_0 = -1$. Since the sum of all basic functions φ_j is identically equal to 1 and the operator $\mathcal{K}_0$ applied to a constant gives a constant, then we may add 1 to $f(\tau)$, denoting $\tilde{f}(\tau) = f(\tau) + 1$; the second differences of $\mathcal{K}_0 \tilde{f}$ are the same. The function $\tilde{f}$ is of form (2) as well, but now $0 \le a_j \le 2$, where $a_0 = 0$. We may suppose that the average value of the function $\tilde{f}$ is not less than $1 - C_* h$, according to Theorem 2.

3. Two different situations are to be considered now: there is or there is not a coefficient a_j greater than γh (where γ is some constant) in a neighbourhood of the gridpoint τ_0.

Theorem 3 (Nearest case). *If*

(a) $\tilde{f}(\tau)$ *is of form* (2), *where* $0 \le a_j \le 2$, $a_0 = 0$;

(b) *at least one of the coefficients* a_j, *corresponding to any gridpoint* $\tau_j \in [-k_* h, k_* h]$, *is greater than* γ, *where* k_* *and* γ *are constants independent on* h;

(c) *the quasiuniform parameter* Q *does not exceed* 23.3,

then for h *small enough there exists such a constant* c *independent on* h *that at least one of inequalities*

$$|D_2(\mathcal{K}_0 \tilde{f}; -2, -1, 0)| > ch, \quad |D_2(\mathcal{K}_0 \tilde{f}; -1, 0, 1)| > ch, \qquad |D_2(\mathcal{K}_0 \tilde{f}; 0, 1, 2)| > ch \tag{7}$$

is valid.

The proof is based on the convexity. As it may be verified, if $Q < 4.25$, then the image $\mathcal{K}_0 \varphi_j$ of each basic function is discretely concave at its "own" gridpoint τ_j and discretely convex at every "alien" gridpoint τ_k; $k \ne j$; now we mean by these words the negativeness or positiveness of the second finite difference on neighbouring three gridpoints τ_{k-1}, τ_k, τ_{k+1}. Moreover, the positive second difference can be estimated from below by $ch^3/|\tau_j - \tau_k|^2$; so it implies the validity of the second inequality in (7). That is because

$$D_2(\mathcal{K}\tilde{f}; -1, 0, 1) = \sum_{j=-N_1}^{N_2} a_j D_2(\mathcal{K}\varphi_j; -1, 0, 1),$$

where every member of the sum is non-negative and one of them satisfies the required estimate.

Unfortunately, if $Q > 4.25$ then the discrete convexity of $\mathcal{K}\varphi_j$ may be violated at one of the two nearest gridpoints. To be more precise, if two ratios $a \equiv h_{3/2}/h_{1/2}$ and $b \equiv h_{-1/2}/h_{1/2}$ are small, then $D_2(\mathcal{K}\varphi_1; -1, 0, 1)$ may be negative.

As to this case, let us modify the criterium of convexity at the gridpoint number 0; namely, we replace the functional $D_2(g; -1, 0, 1)$ by the new functional

$$\mathcal{D}(g) \equiv D_2(g; -1, 0, 1) - pD_2(g; 0, 1, 2). \tag{8}$$

The positiveness of this new functional may be named "generalized convexity" of the function g at the point τ_0. It is possible to establish a connection between the coefficient p in formula (8) and quantities a and b that provides the generalized convexity of every function $\mathcal{K}\varphi_j$; $j \neq 0$ at τ_0. Namely, this connection can be defined by the formula:

$$p = 0.088(1 - a)(1 - b) \cdot \frac{a}{b} \cdot \frac{1 + b}{1 + a}.$$

So the statement of Theorem 3 is true for all $Q < 23.3$.

4. If the coefficients a_j do not exceed any independent constant γ in the neighbourhood of τ_0, then the desired estimate for "local" differences at this central point may be invalid. Then another statement concerning "global" differences may be proved:

Theorem 4. *There exist positive constants k_* and γ independent on h, such that:*

(a) *if $\tilde{f}(\tau)$ is of form* (2), *where $0 \leq a_j \leq 2$; $a_0 = 0$;*

(b) *if for every $\tau_j \in [-k_*h, k_*h]$ the corresponding coefficient $a_j < \gamma$;*

(c) *the average value of $\tilde{f}$ is greater than $1 - C_*h$,*

then for h small enough there exist two gridpoints τ_{-i} and τ_j (where $i, j > 0$) such that the simplified second difference $\tilde{D}_2(\mathcal{K}\tilde{f}; -i, 0, j)$ is greater than ch.

The proof of this theorem is complicated enough.

5. At last the junior term may be returned and the general equation (1) may be considered by the usual methods of the functional analysis. The standard operator $\mathcal{K}_0$ acts from $C^{0,\alpha}$ into $C^{1,\alpha}$. If the junior term is more smooth and acts from $C^{0,\alpha}$ into $C^{2,\alpha}$, then the functional consideration is successful. Finally, we should remark that there was the only point where the restriction for the quasiuniform parameter played its role: it concerns the discrete convexity of the image of the basic function at the neighbouring gridpoint. But, according to numerical experiments, this restriction is not necessary in the problem; this is due to a technique of our consideration only. In order to weaken this restriction or to eliminate it, we can vary the definition of discrete convexity. Namely, we can use as a criterium of convexity the positiveness of some combinations of second differences. This approach is not fulfilled completely.

References

[1] Tsetsokho V.A., Voronin V.V., Smagin S.I. On the solution of diffraction problems by simple layer potentials // Dokl. AN SSSR. – 1988. – Vol. 302, № 2. – P. 323–327 (in Russian).

[2] Voronin V.V., Tsetsokho V.A. Numerical solution of the integral equation of the fist kind with logarithmic singularity // Math. Problems of Geophysics / Computing center. – Novosibirsk, 1973. – Issue 4. – P. 212–228 (in Russian).

[3] Voronin V.V., Tsetsokho V.A. Numerical solution of the integral equation of the 1st kind by interpolation and collocation method // Zhurnal vychisl. matematiki i matem. fiziki. – 1981. – Vol. 21, № 1. –P. 40–53 (in Russian).

[4] Arnold D.N., Wendland W.L. The convergence of spline collocation for strongly elliptic equations // Numerische Mathematik. – 1985. – Vol. 47. – P. 317–341.

[5] Tsetsokho V.A. On the foundation of collocation method for solution the integral equations of the first kind with weak singularities for unclosed curves // Incorrect Problems in Math. Phys. and Anal. – Novosibirsk: Nauka, 1984. – P. 189–197 (in Russian).