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Inverse algorithm for recovering a tsunami source using truncated SVD*

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Abstract. The inverse problem to infer the initial water displacement is treated as an ill-posed problem of the hydrodynamic inversion of tsunami tide gauge records. To this end, we have developed a technique based on the least square inversion and a truncated SVD approach. The proposed filtering technique essentially improves the recover of a tsunami source. We have carried out a series of numerical experiments with synthetic data and a real bathymetry. The accuracy of the tsunami source recovery depends on the spatial distribution of an observation system, target areas and bathymetric features along the wave path. The validity of the approach proposed is confirmed by the numerical results obtained.

Introduction

Tsunamis are rare abrupt marine invasions that cause a severe loss and pain to the coastal communities. The devastating tsunamis have actually put forward their timely warning. One of the most important issues of the tsunami simulation is gaining some insight into a tsunami source. It is known that only some time after an event, having analyzed various seismic, tidal and other data, it appears possible to estimate the basic characteristics of a tsunami source. Thus, the numerical simulation of a tsunami source is one of available tools for the research into tsunami problems. Therefore the interest in the inversion problem for tsunami data has gained as well.0. As is known, the above-mentioned inverse problem is an ill-posed one that stipulates the conditions for all mathematical techniques. There are many different methods and techniques for solving this problem. Some of the studies were based on the assumption that the propagation model is linear. Thus, Satake [1, 2] was the first to propose a formal inversion method for the tsunami waveform, by using Green's functions technique to invert a coseismic slip to a set of simple prescribed rectangular sub-faults from the observed tide gauge data. Tinti et al. [3, 4] have proposed a method where the inversion of tide-gange records to determine the initial waveform was carried out by the least square inversion of a rectangular system of linear equations. One of the main advantages of this method is that it does not require a priori assumption of a fault plane solution: actually, this method

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is completely independent of any particular source model. A similar method for the tsunami waveform inversion is proposed by Pires and Miranda [5] as an alternative to the technique based on Green's functions of the linear long-wave model. Many papers are devoted to estimating the slip distribution of an earthquake in the tsunami center by the inversion of teleseismic body waves.

At first, there was a desire to understand what a minimum number of water level records should be used to recover a tsunami source sufficiently well. For this purpose, we have chosen a linear shallow-water model and an approach based on the SVD and r-solution. The first simple numerical experiments have shown the effectiveness of this approach for the tsunami source function recovery [6].

Based on the typical tsunamigenic earthquakes with a reverse dip-slip or a low-angle trust mechanism and with allowance for the corresponding seismic sources, we numerically simulate the a potential tsunami. The co-seismic deformation of the sea bottom is taken to be identical to the tsunami initial condition.

In the case under study, the initial tsunami waveform problem is formulated as an inverse problem of mathematical physics for restoration of the initial water displacement in the source area by the water-level oscillations observed at a number of points distributed in the ocean. It is well known that the above-formulated problem is an ill-posed one. The possibility to obtain a unique solution exists [7] only when the source function allows factorization, i.e., the dependencies on time and spatial variables are separable. We assume the time dependence to be described by the Heavyside function. A mathematical description of the forward problem of wave propagation consists in a system of linear shallow-water equations in the rectangular coordinates. This system is approximated by the explicit-implicit finite difference scheme on a uniform rectangular grid so that the system of the linear algebraic equations is obtained. The ill-posed inverse problem is regularized by means of the least square inversion using the truncated SVD approach. In this method, the inverse operator is regularized with the help of its restriction on the subspace spanned on a finite number of the first right-hand side singular vectors [6]. The so-called r-solution [8] is a result of the numerical process. The quality of the solution obtained is defined by relative errors of the source function recovery.

In this paper, we make an attempt to answer the following question: how accurately can a tsunami source be reconstructed using records at a given tide gauge network? For answering this question, we have carried out a series of numerical experiments with synthetic data and a real bathymetry. It is necessary to recognize that the results obtained strongly depend on the signal-to-noise ratio due to the ill-posedness of the problem under study. Bearing in mind the fact that a tsunami wave is a low-frequency one in comparison with the background noise, we made an appropriate filtering of calculated signals. The proposed filtering technique essentially improves the recovery of a tsunami source [9] (reference to this paper can be found in [10]). It was found, as in some other methods, that the inversion skill of tsunami sources increases with the improvement of the azimuthal and temporal coverage of assimilated tide gauges stations.

1. Inversion method

Mathematically, the problem of reconstructing the original tsunami waveform in the source area is formulated as determination of spatial distribution of an oscillation source using remote measurements on a finite set of points. One of the most difficult and poorly understood aspects of the tsunami waves propagation is their interaction with the coastline. In this paper we have chosen the simplest of approximate models: the one with a condition of total reflection from a solid wall, consisting in nullifying a normal derivative of the function describing the free surface elevation with respect to the external normal vector. We will neglect the curvature of the Earth. Let us direct the axis z downward. The plane $\{z = 0\}$ corresponds to an undisturbed water surface. Since tsunami in the ocean is a long gravitational wave with a small amplitude, its propagation (for a regional tsunami) can be described by the following linear shallow-water equation

$$W_{tt} = \operatorname{div}(gh(x, y) \operatorname{grad} W) \tag{1}$$

with the initial and boundary conditions

$$W|_{t=0} = \varphi(x, y), \quad W_t|_{t=0} = 0, \tag{2}$$

$$\frac{\partial W}{\partial n}\Big|_H = 0, \tag{3}$$

where $W(x, y, t)$ is a water elevation over the undisturbed state, $h(x, y)$ is the depth of the ocean, g is the acceleration of gravity, $c(x, y) = \sqrt{gh(x, y)}$ is the wave phase speed and n is the unit vector, outwardly directed, normal to the boundary H. The finite function $\varphi(x, y)$ is a bottom displacement in the tsunami center.

Let us solve the problem in the aquatic part of the rectangular domain $\Phi = \{(x; y) : (x_0 \le x \le x_M; \times y_0 \le y \le y_N\}$ on the plane $\{z = 0\}$ with both solid and free marine boundaries. Let Ω be a subdomain of Φ, which is a projection of the bottom displacement domain to the sea surface. It is assumed to be a support of the tsunami center (the target domain). Now, our problem is to recover the bottom uplift $\varphi(x, y)$ in the domain Ω, when the given data, i.e., the water-level elevation $W_0(x, y, t)$, $0 \le t \le T$, are known on a certain set of receivers $R : \{(x_i, y_i) \in G,\ i = 1, \dots, P\}$, disposed on some segment of the line $G : \{(x(s), y(s)),\ 0 \le s \le l\}$ which is a smooth curve without self-crosses $G \in \Phi$.

Let us denote by $\mathcal{A}$ the operator of the solution to the forward problem. We assume that the initial uplift $\varphi(x, y)$ is finite and its support is a compact one and belongs to a limited area of Ω. In addition, the function $\varphi(x, y)$ is assumed to belong to the class $W_2^1(\Omega)$ (the Sobolev space of distributions with square-integrable derivatives). If we assume that the function $h(x, y)$ is continuously differentiable (this assumption does not necessarily correspond to the experiments*), one can assume that the linear operator $\mathcal{A}$ is defined by the following way: for each given $\varphi(x, y)$ one should resolve the Cauchy problem (1)–(3) and trace its solution on the line G. Thus, problem (1)–(3) is now reduced to the following equation:

$$\mathcal{A}\langle\varphi(x, y)\rangle = W_0(s, t), \tag{4}$$

where $\varphi(x, y)$ is the initial bottom elevation, $W_0(s, t)$ is the water elevation on the line G. As was done in [11], the implementation of the standard trace theorem technique gives compactness of the operator $\mathcal{A}$ appointed in (4). As $\mathcal{A} : W_2^1(\Omega) \to L_2((0, L) \times (0, T))$, it does not possess a bounded inverse. The solution of (4) will be sought for in the least-squares formulation:

$$\phi_*(x, y) = \arg\min \|\mathcal{A} < \phi(x, y) > -U(t)\|_{L_2(\mathcal{M}\times(0,T))}.$$

Any numerical method to resolve (4) requires its finite-dimensional approximation. The usual way to do this is projective methods.

The r-solution is a projection of an exact solution to the subspace spanned on the r right-hand side singular vectors corresponding to the top singular values of the compact operator $\mathcal{A}$. It is reasonable that a larger r leads to a more informative solution. Finally, the value r is defined by a singular spectrum and the data noise level. Thus, the numerical solution to equation (4) includes its regularization using the singular value decomposition of the operator $\mathcal{A}$. The operator $\mathcal{A}$ possesses a singular system $\{s_j, \bar{u}_j, \bar{v}_j\}$, where $s_j \ge 0$ $(s_1 \ge s_2 \ge \ldots \ge s_j \ge \ldots)$ are singular values, and $\bar{u}_j$, $\bar{v}_j$ are the left- and the right-hand-side singular vectors. A very important property of singular vectors is that they form a basis in the model and data spaces, so any functions $\varphi(x, y)$ and $W(s, t)$ can be presented as Fourier series:

$$\varphi(x, y) = \sum_{j=1}^{\infty} \alpha_j \bar{v}_j, \quad W_0(t) = \sum_{i=1}^{\infty} W_{0i}\bar{u}_i$$

with $\alpha_j = (\varphi(x, y), \bar{v}_j(x, y))$, $W_{0i} = (W_0(t), \bar{u}_i(t))$. Taking into account these properties, one can construct the r-solution of equation (4) in the form

*Weak solutions in W_2^1 exist and are unique under a much weaker hypothesis on h, for instance, merely that $\log h(x, y)$ be bounded and measurable.

$$\varphi^{[r]}(x,y) = \sum_{j=1}^{r} \frac{(W_0, \bar{u}_j)}{s_j} \bar{v}_j(x,y). \tag{5}$$

This truncated series is stable for any fixed parameter r with respect to perturbations of the right-hand side and the operator itself [8]. As one can see, the ill-posedness of the operator equation of the first kind with a compact operator is due to the fact that $s_j \to 0$ with $j \to \infty$. So, one can perturb the right-hand side $W_0(s,t)$ in such a way that some of its vanishing perturbations can result in rather a large perturbation of the solution. It should be noted that the operator perturbation also brings about the solution instability. The proposed algorithm and substantiation of the validity of this approach are described in considerable detail in [6, 11].

2. Discretization of the problem

Let us assume the domain Ω to be a part of the rectangle $[x_1, x_M] \times [y_1, y_N]$. In order to obtain a system of linear algebraic equations by means of the projective method, in the model space a trigonometric basis was chosen, i.e., the unknown function $\varphi(x,y)$ was sought for as a series of spatial harmonics with unknown coefficients c_{mn}:

$$\varphi(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn} \sin \frac{m\pi}{l_1}(x - x_c) \cdot \sin \frac{n\pi}{l_2}(y - y_c), \tag{6}$$

where $l_1 = (x_M - x_1)$, $l_2 = (y_N - y_1)$, $x_c = (x_1 + x_M)/2$, $y_c = (y_1 + y_N)/2$.

To sample the data, we assume the observation system to consist of P receivers, which are at the points (x_p, y_p), $p = 1, \dots, P$. In this paper, we consider the case when the free surface oscillations $W_0(x_p, y_p, t)$ are known for some finite quantity of the time counts t_j, $j = 1, \dots, N_t$, at each receiver. We introduce a rectangular grid with the step $\Delta x, \Delta y$ over the spatial variables and Δt over the time. System (4) is approximated by the explicit-implicit finite difference scheme on a uniform rectangular grid based on the four-point pattern [12]. The scheme is based on the so-called spatial pattern, which in combination with a central-difference approximation of spatial derivatives simplifies the numerical implementation of boundary conditions, as there is no need to define all the unknown functions on the boundary. The scheme is of second order of accuracy with respect to spatial variables and of first order with respect to time. As was mentioned above, the arrival of the wave at the coast is not considered here.

Simulating the tsunami wave, we need approximation of the two types of boundary conditions: a) conditions on the coastal boundary are assumed to be the full reflection conditions: these are expressed by nullifying the derivative of $W(x,y,t)$ with respect to external normal vector (3); b) conditions on the so-called free boundaries due to an artificial restriction of a considered domain: the absorbing boundary conditions (ABC). In this paper, we use the full absorbing conditions of second order of accuracy [13]. To obtain matrix presentation of the operator $\mathcal{A}$, it is necessary to numerically resolve a forward problem (1)–(3), where each of the basic functions in series (6) is used instead of $\varphi(x, y)$.

3. Numerical experiments: description and discussion

As a model, we used the displacement representing the bottom deformation due to typical tsunamigenic earthquakes with reverse dip-slip or low-angle trust mechanisms. Numerical experiments are presented for the real bottom topography in the case study of the Peru coastal zone. The function to be recovered was chosen in the form

$$\varphi(x, y) = \psi(x, y) \cdot (x - x_0 + 3R_1) \cdot (x - x_0 + R_1/6), \tag{7}$$

where

$$\psi(x, y) = \max\left(1 - \frac{(x - x_0)^2}{R_1^2} - \frac{(y - y_0)^2}{R_2^2},\ 0\right).$$

The domain Φ is a part of the rectangle $\{0 \le x \le 600,\ 0 \le y \le 400\}$ with piecewise-linear boundaries of dry land, the domain $\Omega = \{400 \le x \le 500,\ 200 \le y \le 300\}$ is a rectangle, the center point of the tsunami source was assumed as $(x_c, y_c) = (450, 250)$, $R_1 = 40$, $R_2 = 50$ (all the the horizontal lengths are given in kilometers). In this case, from (7) maximum and minimum values of the function $\varphi(x, y)$ are $\varphi_{\max} = 1.959$ m, $\varphi_{\min} = -0.67$ m, respectively. According (6) $M = 25$ and $N = 11$ were used for modeling the tsunami source (7). Now $\Delta x = \Delta y = 1$, so, the domain Ω contains 100×100 grid points. A series of calculations were carried out by the method proposed and were aimed at recovering the unknown function $\varphi(x, y)$ in (7).

In Figure 1, the contour lines of the real bottom topography of the Peru coastal zone with the target domain Ω and 14 tide gages enumerated clockwise and marked with asterisk $(*)$ are shown.

In Figure 2, typical graphics of common logarithms of singular values of matrix A with respect to their numbers are shown. The curve numbers relate to the quantities of the tide gages used in the recovery procedure. So, we can use $r = 42$ for the case of three receivers (namely, marked with 3, 10, 12) and $r = 115$ for 10 receivers. As practice shows, using $r \le 50$ we can scarcely obtain an acceptable result. In fact, the behavior of graphics depends not only on the number of receivers but on their location and bathymetric characteristics as well.

A sharp decrease in singular values, when their number increases, is typical of all calculations, this is connected with the ill-posedness of the

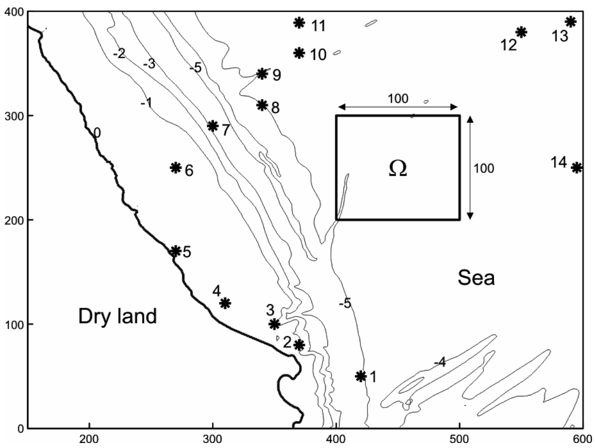

Figure 1

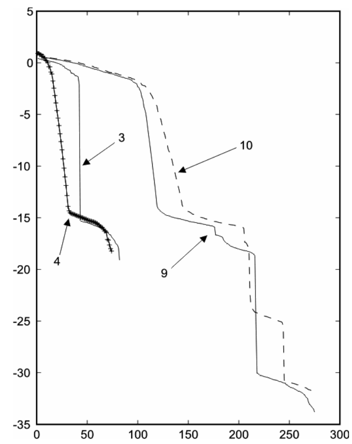

Figure 2

problem. We can choose a parameter r for (5) only from the interval, where the graphics of common logarithms of singular values is slightly sloping. As was mentioned above, matrix A is proceeded from the location in space and the number of tide gages. Thus, comparing the curves for three and four tide gages, one can conclude that a maximum allowable r is influenced not only by the number of tide gauges, but also by their azimuthal coverage.

As was shown in [14], the quality of the tsunami waveform recovery is improved with increasing the number of tide gages, located within the characteristic size of the rectangle of the search and, also, in the case when the receiver captures the most informative signal. After carrying out numerical experiments it becomes clear that the most satisfactory parameter value is $r \geq 70$. In such a way the proposed approach allows one to control a numerical instability of the solution and therefore to obtain an acceptable result in spite of the ill-posedness of the problem. The proposed technique based on the detailed analysis of the properties of the inverse operator makes it possible to obtain a reliable result for a given observation system.

The observed data concerning the form of the arrived wave were simulated as a result of solution to direct problem (1)–(3) perturbed by the background noise, i.e., a high-frequency disturbance. All the experiments discussed in this paper were carried out with the noise rate of 3 % of a signal maximal amplitude over all the receivers. It is necessary to note that the results obtained strongly depend on the presence of noise due to the ill-posedness of the problem. However, since a tsunami wave has much lower-frequency as compared to the background noise, it is reasonable enough to apply the frequency filtration of the observed signal (or synthetic mareograms in as our case).

Influence of the receivers configuration on the recovery accuracy

P	d	r	e	$f_{\max}$	$f_{\min}$	Receivers
7	2	73	0.3767	1.559	−0.7170	3, 4, 5, 6, 9, 10, 11
7	2	55	0.4436	1.381	−0.7680	2, 3, 4, 5, 6, 9, 10
7	2	65	0.4588	1.449	−0.9230	5, 6, 7, 8, 9, 10, 11
7	3	93	0.3164	1.732	−0.6472	5, 6, 7, 8, 9, 10, 11
7	4	99	0.2671	1.816	−0.7028	5, 6, 7, 8, 9, 10, 11
7	2	42	0.5520	1.276	−0.6745	1, 3, 5, 7, 9, 11, 13
7	4	59	0.3173	1.706	−0.7371	1, 3, 5, 7, 9, 11, 13
7	2	23	0.6278	1.106	−0.6959	3, 4, 5, 6, 7, 8, 9
7	2	45	0.5752	1.431	−1.3500	1, 2, 3, 5, 8, 9, 10
7	2	66	0.6443	1.995	−1.9200	1, 2, 3, 5, 9, 10, 11

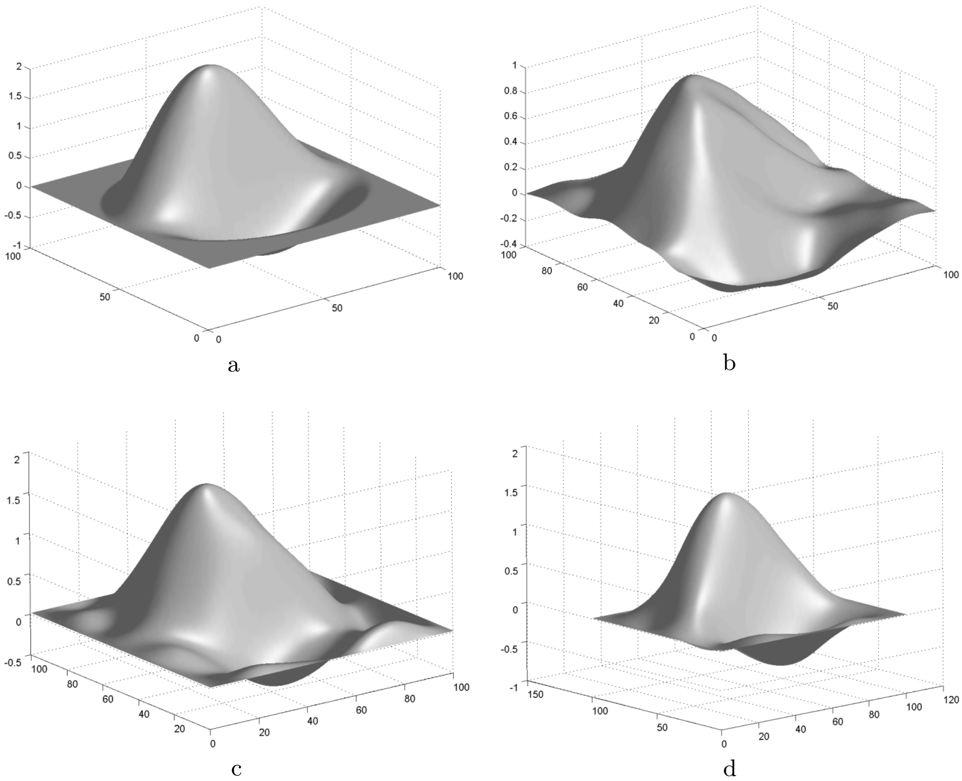

Figure 3. Wave forms: a) initial — $\varphi_{\max} = 1.959$, $\varphi_{\min} = -0.67$; b) recovered with three tide gages — $\varphi_{\max} = 1.213$ (0.885), $\varphi_{\min} = -0.738$ (−0.357), $r = 41$, $e = 0.717$; c) recovered with five tide gages — $\varphi_{\max} = 1.757$ (1.4716), $\varphi_{\min} = -1.0073$ (−0.5142), $r = 57$, $e = 0.4729$; d) recovered with seven tide gages — $\varphi_{\max} = 1.835$ (1.5138), $\varphi_{\min} = -0.7016$ (−0.5484), $r = 103$, $e = 0.262$, $d = 6$

The initial data smoothing was performed by using an original method of the grid function smoothing proposed at the Computing Center of the Siberian Branch of USSR Academy of Sciences in 1974 [9] (see also [10]). The idea of the original algorithm is in sewing n-times differentiable local approximations of the grid function with the help of Partition of Unity Method. As the recovered tsunami source is characterized by essential noise, which can cause a considerable distortion by the numerical modeling of tsunami propagation and tsunami inundation, we made use of an appropriate 2D-smoothing procedure (based on the above method) for the recovered tsunami waveform as well. The filtering technique proposed essentially improves the results of recovery.

In these calculations, r-solution was obtained with cond ≥ 1000. The table gives an idea of the influence of the receivers configuration on the recovery accuracy. Here P is the number of receivers in configuration, d is $\log(1/\operatorname{cond})$, e is a relative error (in L_2-norm) in the source function, $f_{\max}$ and $f_{\min}$ are extremal values of the source function, and the last column represents lists of receivers.

As a result of the numerical simulation, we have obtained the initial water displacement in the tsunami source area. In Figure 3, the initial and the recovered tsunami waveforms are presented. The values in parentheses correspond to smoothing the waveforms. The quality of the solution strongly depends on the number of receivers and their disposition and was evaluated as relative errors (in L_2-norm) in the source function (before smoothing).

After smoothing the recovered waveform, we solved the direct problem with the tsunami waveform obtained. We calculated mareograms at the same points and compared them with the synthetic ones, which were considered as initial data in our recovering. One can see them in Figure 4.

Some of them relate to the receivers used in recovering but others — to the rest observational points. The solid line is to exact (synthetic) mareograms, the dashed line is to mareograms for five receivers (3, 4, 6, 7, 10) and a dash-dotted line is to for seven receivers (5–11). As is seen in this case, the calculations with five receivers do not provide a sufficient match of mareograms. But seven receivers give a good agreement both in receivers used in the algorithm and at other observational points. If the main purpose is the prediction of water elevation in the area based on the data provided by some tide gauges, the agreement of mareograms is more important for us than the accuracy in the form of a tsunami source.

Conclusion

This paper proposes a new approach to the problem of recovering a tsunami source, followed by the calculation of water elevation in the area in question. This method allows us to appreciate a minimal number of receivers needed

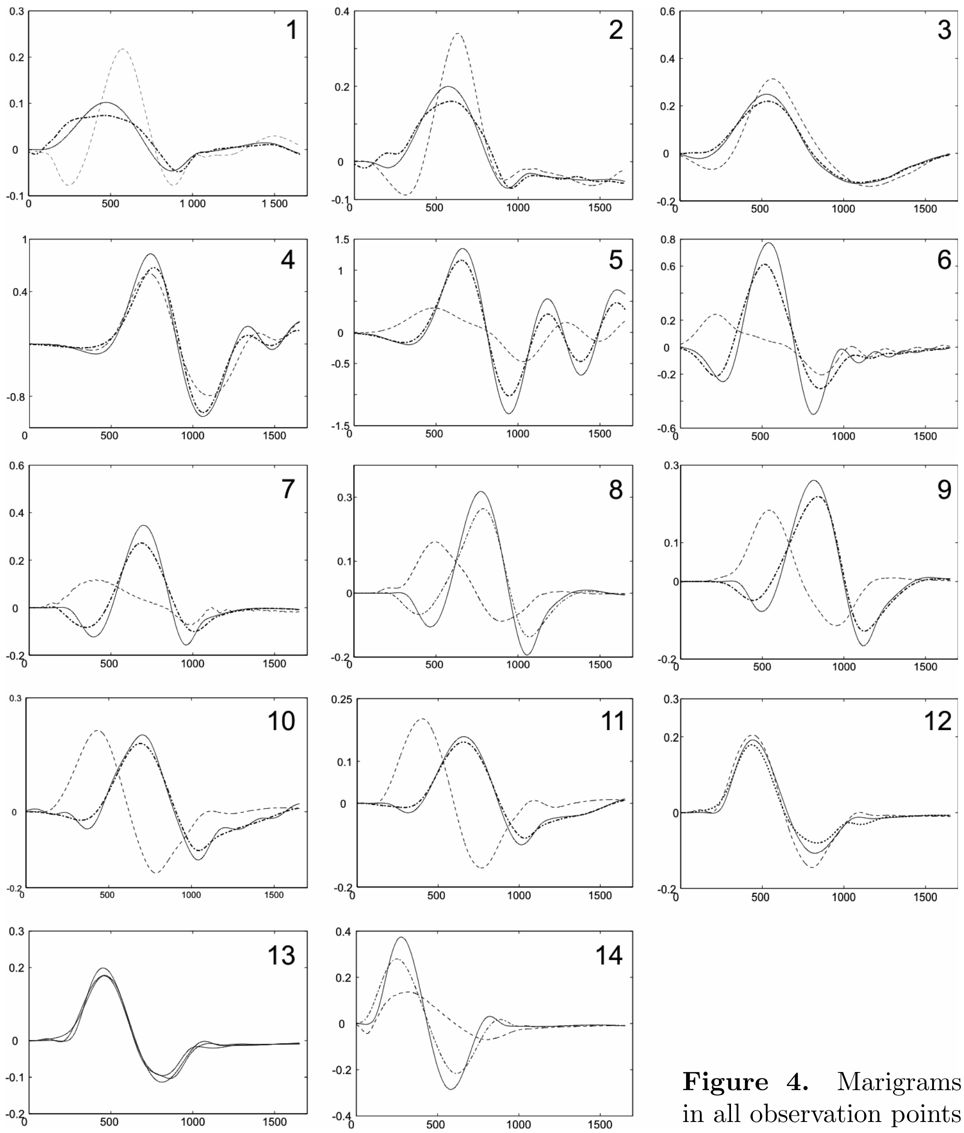

Figure 4. Marigrams in all observation points

for the tsunami source recovery. This approach also allows us to control numerical instability and therefore to obtain an acceptable result in spite of the ill-posedness of the problem. The SVD technique and the usage of r-solutions are combined with smoothing filtering procedures both the initial data and the calculated tsunami waveform. This leads to an essential improvement in calculated water elevations. Such an approach proves to be extremely helpful to evaluate a real possibility of a given tide gauges system for recovering the initial water displacement.

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