Bull. Nov. Comp. Center, Math. Model. in Geoph., 12 (2008), 83–91

Numerical simulation of a tsunami source: the case study of the Peru coastal area*

T.A. Voronina

Abstract. This paper studies the importance of spatial distribution of the sea-level monitoring stations for the accuracy of restoration of a tsunami source. The ill-posed inverse restoration problem is regularized by means of the least squares inversion using a truncated SVD approach. The wave propagation is described by linearized shallow-water equations when depth depends on an arbitrary function of two variables. Results of numerical experiments are presented in the case study of the Peru coastal area.

1. Introduction

Recently, the devastating tsunamis have acutely put forward the problem for their timely warning. People living in the costal areas, for example, on the South American Pacific coast, are always exposed to tsunami risks, because abrupt marine invasions such as tsunamis are particularly devastating for the coastal areas. False tsunami warnings are a cause of great financial losses. Scientists and engineers are joining forces for a more accurate prediction of tsunami dangers. An important component of the assessment and, thus, mitigation of the effects of the tsunami impact is computer-aided simulation. One of the main needs for developments in tsunami modeling is estimating the characteristic parameters of a tsunami source. Most of these parameters can only be estimated by compilation of seismic, geophysical and tidal data some time after the event. Thus, the numerical modeling of a tsunami source is one of available tools for the research into tsunamis problems. The recover of a tsunami source requires the "inversion" of a marigram recorded at monitoring stations. The accuracy of restoration of a tsunami source depends on a spatial distribution of a monitoring system, respectively, the target areas and bathymetric features along the way path. For this reason, we have developed a technique based on the inverse problem by a truncated SVD approach.

In this paper, we make an attempt to answer the following questions:

- How accurately can we restore the source of tsunami in marigrams recorded at the existing system of observation?

*Supported by the Russian Foundation for Basic Research under Grants 08-07-10105, IG 2006-113.

- Is it possible to improve the recovery by providing the “most informative” direction to the monitoring system?
- Can accurate source models be developed in the near-real time?

With a view towards answering these questions, we have proposed a new approach of the so-called dynamic tsunami source modeling. The proposed methodology is validated with experiments using synthetic data and real bathymetry. This approach includes the following steps. First, we obtain the synthetic tide gauge records from a model source, whose form we are to reconstruct. These can be records observed at real time instants. The original tsunami source in the area in question is recovered by the inversion of the above wave records. We calculate mariograms from the earlier reconstructed source. To define the “most informative” part of the initial observation system for a target area, we compare synthetic mariograms, obtained in two cases in the same locations (it will compare synthetic and real recordings) at all available sea-level tide-gauges. Next, we consider the observation system, which contains only good matching stations. Now we can again restore the tsunami source using only the tide-gauge records that were determined as being the “most informative” part. This “improved” tsunami source can be proposed for the use in further tsunami calculation. Taking into account a prior information that the tsunami is a long gravitational low-amplitude wave at all above stages, an appropriate filtering of the data and calculated signals was made. It was found, as in other methods, that the inversion skill of a tsunami source increases with azimuthal and temporal coverage of assimilated tide-gauges stations.

2. Restoration of the initial tsunami waveform

Mathematically, the problem of reconstructing the original tsunami waveform in the source area is formulated as determination of spatial distribution of an oscillation source using remote measurements on a finite set of points. Thus, this problem is implemented by the inversion of the wave records observed on a set of the sea-level stations – the “data space”. The “model space” is represented by a linear combination of given basic functions.

The mathematical description of the direct problem of the wave propagation consists in a linear shallow-water system of differential equations in the rectangular coordinates:

$$W_{tt} = \operatorname{div}(h \operatorname{grad} W) \tag{1}$$

with the initial and the boundary conditions:

$$W|_{t=0} = \varphi(x, y), \quad W_t|_{t=0} = 0, \tag{2}$$

$$\left.\frac{\partial W}{\partial n}\right|_L = 0, \tag{3}$$

where $W(x,y,t)$ is a water elevation over the undisturbed state, $h(x,y)$ is a depth of the ocean, $f(x,y,t)$ describes the movement of the bottom in the tsunami area. The velocity of the tsunami wave propagation is also described as $c(x,y) = \sqrt{gh}$. A unique solution exists only when the function of the source allows factorization [1], i.e., the function $f(x,y,t)$ can be factorized as $f(x,y,t) = \varepsilon(t) \cdot \varphi(x,y)$, where $\varepsilon(t)$ is the Heaviside function, $\varphi(x,y)$ is shearing of bottom in the tsunami center. We solve the problem in the domain Φ with piecewise-linear inner and outer boundaries. Let Ω be a subdomain of Φ, and it is assumed to be a support of the tsunami center (the target domain).

Now, our problem is to recover the bottom displacement $\varphi(x,y)$ in the domain Ω, when the given data, i.e. the water elevation $W_0(x,y,t)$, $0 \le t \le T$, are known on a certain set of the receivers $M = \{\,(x_i,y_i) \in G$, $i = 1,\ldots,P\}$, disposed on some segment of the line G, this may be a smooth curve without self-crosses, $G = \{((x(s),y(s)),\ 0 \le s \le l\}$.

Let us denote by $\mathcal{A}$ the operator of the solution to the direct problem. Thus, the solution to problem (1)–(3) is now reduced to the vector equation

$$\mathcal{A}\langle\phi(x,y)\rangle = \bar{W}_0(x_i,y_i,t), \tag{4}$$

where $\phi(x,y)$ is the initial bottom elevation, $\bar{W}_0(x_i,y_i,t)$ is water elevation on the line G. One can assume that the linear operator $\mathcal{A}$ is defined by the following way: for each given $\phi(x,y)$ one should resolve the Cauchy problem (1-3) and trace its solution at the points from the set M. Let us assume that the function $\phi(x,y)$ is of the class ${W_2}^1(\Phi)$, the function $h(x,y)$ is continuously differentiable. According to [2], we can consider

$$\mathcal{A} : {W_2}^1(\Phi) \to L_2((0,L)\times(0,T)).$$

As was done in [4], it is possible to prove that this operator $\mathcal{A}$ is a compact one, so it does not possess a bounded inverse. Any attempt to numerically resolve equation (4) should be followed by some regularization procedure. In the present paper, we use an inversion method, already described in the previous publications [3]. The inverse operator is regularized with the help of its restriction on a subspace spanned on a finite number of the first right singular vectors [4]. Any compact operator possesses a singular system $\{s_j,\bar{u}_j,\bar{v}_j\}$, i.e., the singular values $s_j \ge 0$ $(s_1 \ge s_2 \ge \ldots \ge s_j \ge \ldots)$ and the left $(\bar{u}_i)$ and the right $(\bar{v}_j)$ singular vectors. A very important property of singular vectors is that they form a basis in the model and the data spaces. As one can see from [6], the ill-posedness of the operator equation of the first kind with a compact operator is due to the fact that $s_j \to 0$ with $j \to \infty$. So, one can perturb the right-hand side $W_0(x_i, y_i, t)$ in such way that some its vanishing perturbation $\varepsilon(t)$ can result in rather a large perturbation of the solution. For example, $\varepsilon(t) = \varepsilon_j W_0(x_i, y_i, t)$, with $\varepsilon_j \to 0$ as $j \to \infty$ in such a way that $s_j/\varepsilon_j \to 0$. It should be noted that the operator perturbation also brings about the solution instability.

The numerical solution to equation (4) includes its regularization using the SVD-decomposition of the operator $\mathcal{A}$, that leads to constructing r-solution given by the relation

$$\phi_{[r]}(x, y) = \sum_{j=1}^{r} \frac{(\bar{W}_0, \bar{u}_j)}{s_j} \bar{v}_j(x, y). \tag{5}$$

This truncated series is stable for any fixed parameter r with respect to perturbations of the right-hand side and the operator itself [5]. System of equations (4) is approximated by the explicit-implicit finite difference scheme on a uniform rectangular grid based on the four-point pattern [7]. The scheme is of second order of accuracy with respect to spatial variables and of first order with respect to time.

Let us assume Φ to be a subdomain of the rectangle $\Pi = \{x_0 \le x \le x_M, y_0 \le y \le y_N\}$. A uniform rectangular grid is defined in Π, but in fact, part of grid points correspond to the dry land. So, the difference scheme employs only the grid points disposed in Φ. The arrival of the wave to the coast is not considered here. Simulating the tsunami wave, we need approximation of the two types of boundary conditions: a) conditions on the coastal boundary are assumed to be the full reflection conditions—these are expressed by nullifying the derivative of $W(x, y, t)$ with respect to the external normal vector (3); b) conditions on the so-called free boundaries due to an artificial restriction of a considered domain—the absorbing boundary conditions (ABC). In this paper, we use the full absorbing conditions of second order of accuracy [8].

3. Numerical experiments: description and discussion

Let us assume the domain Ω to be on $[x_1, x_M] \times [y_1, y_N]$ rectangle. In order to obtain a system of linear algebraic equations by means of the projective method, in the model space, the trigonometric basis was chosen, i.e., an unknown function $\phi(x, y)$ was sought for as a series of spatial harmonics with unknown coefficients c_{mn}:

$$\phi(x, y) = \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn} \sin \frac{m\pi(x - x_c)}{l_1} \sin \frac{n\pi(y - y_c)}{l_2}, \tag{6}$$

where $l_1 = x_M - x_1$, $l_2 = y_N - y_1$, x_c and y_c are central points of the intervals.

To sample the data, we suppose the observation system to consist of P receivers, which are at the points (x_p, y_p), $p = 1, \dots, P$. We consider the case when the free surface oscillations $W_0(x_p, y_p, t)$ (marigram) are known for some finite quantity of the time counts t_j, $j = 1, \dots, N_t$, in each receiver. It will be reasonable to choose a basis in the data space as vectors $\vec{u}^{(kj)}$, $k = 1, \dots, P$, $j = 1, \dots, N_t$, with components $(u_i^{kj}) = \delta_{ij}$. So, the dimensions of the solution and of data spaces are $\dim_{\text{sol}} = K = M \times N$; $\dim_{\text{data}} = L = P \times N_t$. This leads to the following system of linear algebraic equations with respect to unknown coefficients $\{c_{nm}\}$:

$$\boldsymbol{A}\tilde{c} = \tilde{f}, \tag{7}$$

where

$$\begin{gathered}\tilde{f} = (w_{11}, \dots, w_{1N_t}, w_{21}, \dots, w_{2N_t}, w_{P1}, \dots, w_{PN_t})^T;\\ w_{ij} = W_0(x_i, y_i, t_j), \quad i = 1, \dots, P, \quad j = 1, \dots, N_t;\\ \tilde{c} = (c_1, c_2, \dots, c_k)^T.\end{gathered}$$

To obtain the matrix $\boldsymbol{A}$, it is necessary to numerically resolve direct problem (1)–(3), where instead of $\phi(x, y)$ the basic function is used. The matrix $\boldsymbol{A}$ is a rectangular one and possesses SVD decomposition $\boldsymbol{A} = \boldsymbol{U\Sigma V}^T$, where $\boldsymbol{V}$ is $K \times K$ matrix and $\boldsymbol{U}$ is $L \times L$ matrix. The right-hand singular vectors $\tilde{v}_j$ of the matrix $\boldsymbol{A}$ are columns of the matrix $\boldsymbol{V}$, they make a basis in the space of solutions. The left-hand singular vectors $\tilde{u}_i$ are columns of the matrix $\boldsymbol{U}$ and they make a basis in the space of the right-hand side. So, according to (5), one should find r-solution to a finite-dimensional approximation as

$$\phi_{[r]^{KL}}(x, y) = \sum_{j=1}^{r} \frac{(\tilde{f}, \tilde{u}_j)}{s_j} \tilde{V}_j(x, y), \tag{8}$$

where $\tilde{V}_j(x, y) = \sum_{k=1}^{K} v_{jk}\phi_k(x, y)$ and $r = \max\Big\{k : \frac{s_k}{s_1} \geq \frac{1}{\text{cond}}\Big\}$.

It turned out that the number r of basic vectors applied appears to be essentially lower than a minimum dimension of a matrix. The number r depends both on a singular spectrum of the matrix $\boldsymbol{A}$ and on the noise level in the observed signals. The singular spectrum of matrix $\boldsymbol{A}$ is determined by the monitoring system. This dependence is investigated in the present paper with the use of numerical simulation. The quality of the solution strongly depends on the number of receivers and their disposition and is evaluated as relative errors (in L_2-norm) in the source function restoration.

Such an approach proves to be extremely helpful to evaluate a real possibility of a given tide-gauges system for restoration of the initial water displacement. As a model we used the displacement representing the bottom deformation due to typical tsunamigenic earthquakes with reverse dip-slip or low-angle trust mechanisms. Numerical experiments are presented for the model bottom topography in the case study of the Peru coastal zone.

The function to be recovered was chosen in the form $\varphi(x,y) = \psi(x,y) \cdot \alpha(x)$, where $\alpha(x)$ depends on the tips of a model, in our case, $\alpha = (x - x_0 + 3R_1)(x - x_0 + R_1/6)$, and the function $\psi(x,y)$ describes the paraboloid

$$\psi(x,y) = \max\left\{1 - \frac{(x-x_0)^2}{R_1^2} - \frac{(y-y_0)^2}{R_2^2},\ 0\right\}.$$

The domain $\Pi = \{0 \le x \le 600,\ 0 \le y \le 400\}$ is a rectangle, the domain $\Omega = \{400 \le x \le 500,\ 200 \le y \le 300\}$ is a rectangle, the center point of the tsunami source is $(x_c, y_c) = (450, 250)$, $R_1 = 40$, $R_2 = 50$ (all the sizes are measured in kilometers). Therefore, $\varphi_{\max} \approx 1.959$ m; $\varphi_{\min} \approx -0.67$ m. The function $\varphi(x,y)$ was approximately found in the form of (8), where $M = 25$, $N = 11$. A series of calculations were carried out by the method proposed and were aimed at recovering the unknown function $\varphi(x,y)$.

In all these calculations, r-solution was obtained when cond $= 10$. In Figures 1 and 2, the bottom topography and common logarithms of singular values of the matrix $\boldsymbol{A}$ are shown. A sharp decrease in singular values, when their number increases, is typical of all the calculations. The influence of the conditioning number is essential, too. However, significant oscillations appear in the solution when the conditioning number cond > 10. This is typical for all ill-posed problems. The observed data concerning the form of the arrived wave were simulated as a result of solution to direct problem (1)–(3), perturbed by the background noise, i.e., a high-frequency disturbance. All experiments presented here were carried out with the disturbance rate of 3 % of a maximum amplitude of a signal over all receivers. It is necessary to note that the results obtained strongly depend on the presence of disturbance due to the ill-posedness of the problem. However, since the tsunami wave is of an essentially lower frequency as compared to the background noise, it is

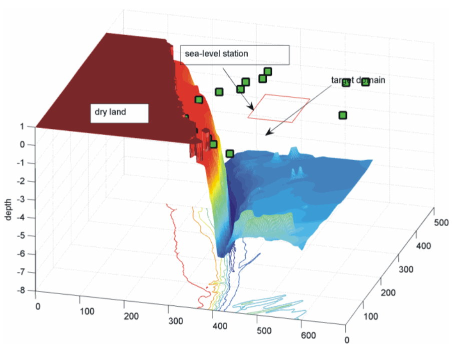

Figure 1. Bottom topography

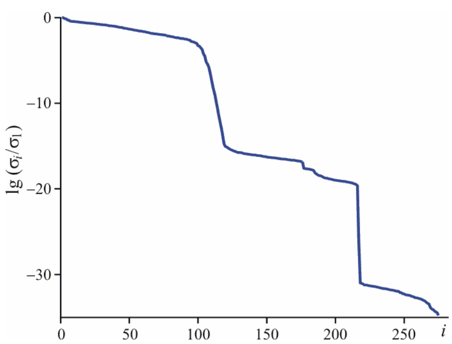

Figure 2. Singular values of matrix $\boldsymbol{A}$ in common log scale

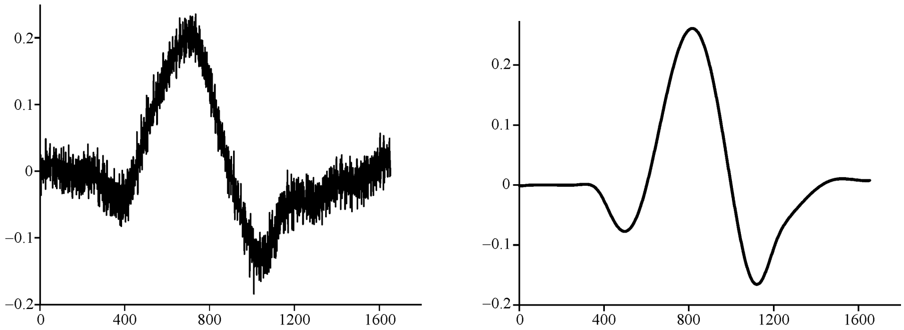

Figure 3. Typical signals with noise of 3 % and the same signals after filtration

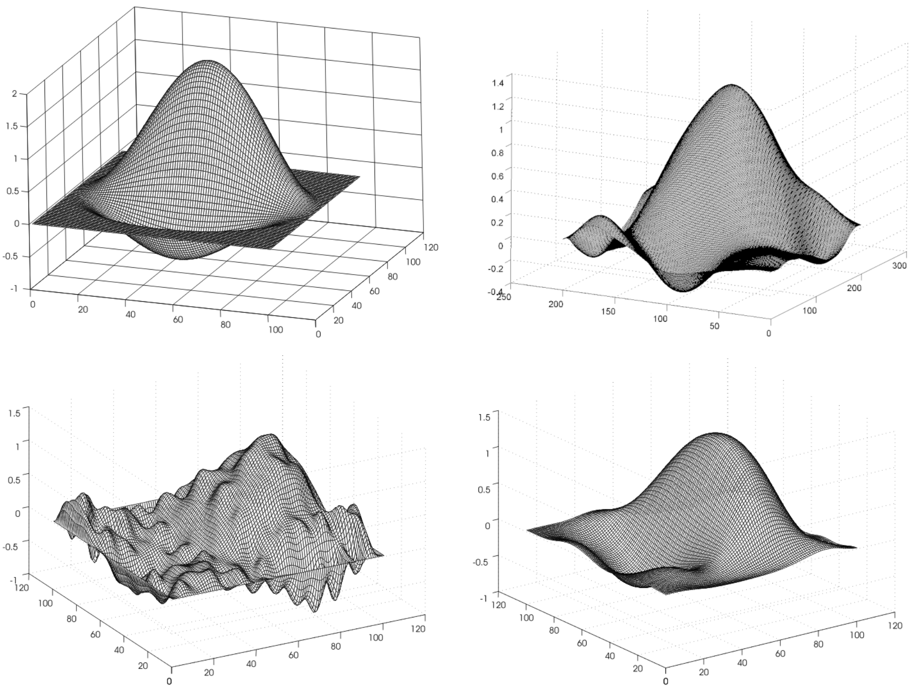

Figure 4. Initial, corrected (with nine observation stations), recovered, and corrected (with six observation stations) tsunami waveforms

reasonable enough to apply the frequency filtration of the observed signal.

In this paper, the filtration is done by the method described in [9]. In Figure 3, there are typical signals with noise and the same signals after filtration. As a result of the numerical simulation, we have obtained the initial water displacement in the tsunami source area. In Figure 4, the initial and the recovered tsunami waveforms are represented for the recovered tsunami waveform $\varphi_{\max} \approx 1.399$ m, $\varphi_{\min} \approx -0.47$ m, and for the "corrected" wave-

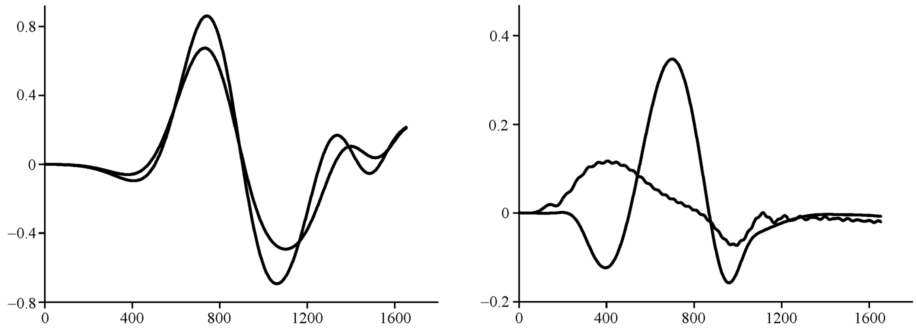

Figure 5. Synthetic marigrams from initial and recovering tsunami sources at the fifth and the eighth points of the given monitoring system

form — $\varphi_{\max} \approx 1.328$ m, $\varphi_{\min} \approx -0.5$ m, in the case of six observed points; $\varphi_{\max} \approx 1.549$ m, $\varphi_{\min} \approx -0.6591$ m for the nine observed points, respectively.

One can see that the recover of the tsunami source was characterized by an essential noise. Probably, the numerical models of tsunami propagation and tsunami inundation cannot be adequately calculated in this case. This means that the recovering tsunami waveform was filtered over as well as the "corrected" tsunami source (see Figure 4). Thus, using the "corrected" tsunami source, we calculated marigrams at the same points and compared them to the observed ones. For some points of the observed system, it was a good match, for the others — it was not (Figure 5). We believe that one of the reasons of a poor coincidence of marigrams is bathymetric features of the bottom relief. After that, we reformed the exiting observation system — we kept only the first type points in the system. So, we have considered an appropriate observation system and a tsunami source in a certain area to use in the further forecast of the tsunami risk. Thus, we have obtained the initial tsunami waveform matching the subsequent research.

4. Conclusion

This paper proposes a new approach to solving the problem of restoring a tsunami source. We have developed the technique based on the inverse problem by a truncated SVD approach. The proposed technique for filtering both the initial data and calculated tsunami waveforms essentially enhances the numerical results, which were found to be highly sensitive to a spatial distribution of the monitoring stations relative to bathymetric features.

References

[1] Kaistrenko V.M. Inverse problem for reconstruction of tsunami source // Tsunami Waves. Proc. Sakhalin Compl. Inst. — 1972. — Issue 29. — P. 82–92.

[2] Ladiejenskaya O.A. Boundary-value problems of mathematical physics. — Moscow: Nauka, 1973.

[3] Voronina T.A., Tcheverda B.A. Reconstruction of tsunami initial form via level oscillation // Bull. Novosibirsk Comp. Center. Ser. Math. Model. in Geoph. — Novosibirsk, 1998. — Iss. 4. — P. 127-136

[4] Voronina T.A. Determination of spatial distribution of oscillation sources by remote measurements on a finite set of points // Sib. J. Numer. Math. — 2004. — Vol. 7, No. 3. — P. 203–211.

[5] Cheverda V.A., Kostin V.I. R-pseudoinverse for compact operators in Hilbert space: existence and stability // J. Inverse and Ill-Posed Problems. — 1995. — Vol. 3, No. 2. — P. 131–148.

[6] Zuhair Nashed M. Aspects of generalized inverses in analysis and regularization // Generalized Inverses and Applications. — 1988. — P. 193–244.

[7] Marchuk An.G., Chubarov L.B., Shokin Yu.I. The Numerical Simulation of the Tsunami Wave. — Novosibirsk: Nauka, 1983.

[8] Enquist B., Majda A. Absorbing boundary conditions for the numerical simulation of waves // Math. Comp. — 1977. — Vol. 31, No. 139. — P. 629–654.

[9] Belonosov A.S., Tsetsokho V.A. The numerical algorithm and procedure of smoothing functions, approximate given at grid nodes of a irregular mesh on the plane // Some Problems of Math. Physics and Interpretation of the Geophysical Observations. — Computing Center SB AS USSR, 1976. — P. 6–29.