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Reconstruction of the initial tsunami waveform by the coastal observations inversion

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The paper deals with the reconstruction of the original tsunami waveform in the source area records from the waves observed on a set of the coastal wave gauges. The wave propagation is described by the linearized shallow-water equations when depth depends on one variable. The direct problem is approximated by the explicit-implicit finite difference scheme. The ill-posed inverse problem of the reconstruction is regularized by means of the least square inversion using the truncated SVD approach, so r-solution is a result of the numerical process. Results of numerical experiments are presented.

1. Introduction

This paper is the sequel and further development of the earlier proposed approach [10], which concerns the reconstruction of the original tsunami waveform when receivers are disposed at some points on the coast. In [10], the consideration was limited to the case of the flat bottom, in the present paper, the depth depends on one variable that is the distance from the coast. This model corresponds to the bottom topography of the Kuril–Kamchatka zone. We may reasonably consider that our conclusions do not variate very much when the general case of an arbitrary bottom is considered.

The problem of the reconstruction is posed as an inverse problem of the mathematical physics. To solve this ill-posed problem we should apply the approach using the so-called r-solutions such as in [6]. In this method, the inverse operator is regularized with the help of its restriction on the subspace spanned on a finite sample of the first right singular vectors.

As for a general case, the problem of the reconstruction of a source is not uniquely solvable (see [1]). The uniqueness is provided by factorization of the function which describes the action of this source, namely, the dependence on time and spatial variables can be separated. Furthermore, the dependence on time is *a priori* given.

The mathematical description of the direct problem of the wave propagation consists in the linearized shallow-water system of differential equations written in the rectangular coordinates. This system is approximated by the explicit-implicit finite difference scheme on a uniform rectangular grid, so the system of the linear algebraic equations is obtained. Then the SVD-analyses is applied to its matrix, and a generalized r-solution is sought. Its quality depends on the number of receivers and their spatial disposition. This dependence is investigated in the present paper by means of the numerical simulation.

2. Statement of the problem

The wave propagation is described within the framework of the linear shallow water theory, i.e., by a usual scalar wave equation for the water elevation (variation in wave velocity is connected only with the bottom topography), see [1]. Let us neglect the curvature of the Earth. Let the z-axis be directed down-awards with depth. The plane $\{z = 0\}$ corresponds to undisturbed water surface. Since the tsunami wave in the ocean is a long gravitational wave with a small amplitude, its propagation can be described by the shallow water equation:

$$W_{tt} = \operatorname{div}(h(x,y)\operatorname{grad} W) + f_{tt}(t,x,y) \tag{1}$$

with the initial conditions

$$W|_{t=0} = 0; \quad W_t|_{t=0} = 0 \tag{2}$$

and the boundary conditions

$$W|_{\Gamma} = W_0(x(s), y(s), t), \tag{3}$$

where $W(x,y,t)$ is a water elevation over the undisturbed state, $h(x,y)$ is the depth of the ocean, $f(x,y,t)$ describes the movement of the bottom in the tsunami source area. In this paper, we consider a depth as a function of a spatial variable only: $h(x,y) = h(x)$. Then the velocity of the tsunami wave propagation is also described as $c(x,y) = c(x) = \sqrt{gh}$.

We assume the function $f(x,y,t)$ describing the movement of the bottom in the source area can be factorized as follows: $f(x,y,t) = \varepsilon(t)\cdot\varphi(x,y)$, where

$$\varepsilon(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0 \end{cases}$$

is the Heavyside function.

Under the above assumptions, equation (1) can be replaced by the linearized shallow water equations

$$U_t + gW_x = 0, \quad V_t + gW_y = 0, \quad W_t + (hU)_x + (hV)_y = 0 \tag{4}$$

with the boundary conditions

$$W|_{\Gamma} = W_0(x(s), y(s); t) \tag{5}$$

and the initial conditions

$$W|_{t=0} = \varphi(x, y), \quad U|_{t=0} = V|_{t=0} = 0, \tag{6}$$

where $W(x, y, t)$ is the elevation of the wave over the ocean level, $(U(x, y, t), V(x, y, t))$ is a vector of the velocities and g is the gravity constant.

Simulating the tsunami wave with the help of the system (4) with the boundary and initial conditions (5) and (6), we need approximation of the two kinds of the boundary conditions:

a) the conditions on the coastal boundary are assumed to be the full reflection conditions; these are expressed by vanishing when the derivative of $W(x, y, t)$ with respect to the external normal vector:

$$\left.\frac{\partial W}{\partial n}\right|_{x=0} = 0; \tag{7}$$

b) the conditions on the so-called free boundaries due to the artificial restriction of a considered domain; these are expressed by the conditions of free passing and the absorbing conditions. In this paper, we use the full absorbing conditions of the second degree of accuracy [3].

We solve the problem in the rectangular domain $\Phi = \{(x, y) : \ x_0 \leq x \leq x_M, \ y_0 \leq y \leq y_N\}$. Let us assume that y-axis is directed along the coastal line, and x-axis is perpendicular to it.

The Ω denotes the subdomain of Φ which is assumed to be a support of the tsunami center (the target object domain).

Now our problem is in recovering the bottom movement $\varphi(x, y)$ in the domain Ω, when the given data is the water elevation $W_0(x, y, t)$ in a certain set of the receivers $\{M_i = (x_i, y_i), \ i = \overline{1, P}\}$, disposed on the boundary Γ:

$$W_0(x, y; t)|_{M_i} = w_{0i}(t), \quad 0 \leq t \leq T.$$

In this work, we consider the case when the free surface oscillations are known on some segment of the boundary coinciding with the y-axis (this is the coast line), though this method is also applicable for an arbitrary form of the boundary (this may be a smooth curve without self-crosses).

Let us assume that the support of the function $\phi(x, y)$ is included in the rectangle Φ and this function is of the class $W_2^1(\Phi)$.

So, that the solution of problem (4)–(6) is now reduced to the following vector equation:

$$\mathcal{A}\langle\phi(x, y)\rangle = U(t) = \begin{pmatrix} w_{01}(t) \\ w_{02}(t) \\ \vdots \\ w_{0P}(t). \end{pmatrix}. \tag{8}$$

We can consider $\mathcal{A} : W_2^1(\Phi) \to L_2(M \times (0,T))$. As it was done in [10], it is possible to prove that this operator is a compact one, so it does not possess a bounded inverse. The numerical solution to equation (8) includes its regularization using the SVD-decomposition of the operator $\mathcal{A}$, that leads to the construction of r-solution [5]. The algorithm used in this way and substantiation of using this approach are described in detail in [10].

3. Discretization of the problem

Let us assume that the domain Ω is $[x_b, x_e] \times [y_b, y_e]$ rectangle. The unknown function $\varphi(\xi, \zeta)$ will be sought in the form of a series of spatial harmonics with the unknown coefficients a_{mn}, $m = 1, 2, \dots, n_x$, $n = 1, 2, \dots, n_y$, namely:

$$\varphi(\xi, \zeta) = \sum_{m=1}^{n_x} \sum_{n=1}^{n_y} a_{mn} \sin \frac{m\pi}{l_1}(\xi - x_b) \cdot \sin \frac{n\pi}{l_2}(\zeta - y_b), \tag{9}$$

where $l_1 = (x_e - x_b)$, $l_2 = (y_e - y_b)$. Let us discretize the observed wavefield. The observation system is assumed to consist of k_p receivers, their coordinates being (x_p, y_p), $p = 1, \dots, k_p$, the mareogram being known (the wave elevation as the function of time) in each of them, i.e., $W_0(x_p, y_p, t)$ are given. The boundary Γ of the domain Φ consists of four straight segments numbered by the Roman numbers I–IV respectively: $\{x = 0\}$, $\{y = y_N\}$, $\{x = x_M\}$, and $\{y = 0\}$.

For the discretization in the problem we introduce the rectangular grid with the step Δx, Δy in spatial variables and Δt with respect to time:

$$\begin{aligned} x_i &= x_0 + i\Delta x, \quad i = 1, \dots, k_x, \quad \text{where } \Delta x = (x_M - x_0)/k_x, \\ y_j &= y_0 + j\Delta y, \quad j = 1, \dots, k_y, \quad \text{where } \Delta y = (y_N - y_0)/k_y, \\ t_k &= t_0 + k\Delta t, \quad k = 1, \dots, k_t, \quad \text{where } \Delta t = (t_k - t_0)/k_t. \end{aligned}$$

The system of equations (4) is approximated by the explicit-implicit finite difference scheme on the uniform rectangular grid based on the 4-point pattern [7]. The scheme is of second order of accuracy with respect to spatial variables and of the first order with the respect to time. The scheme is derived applying the method described in [8]. The scheme is stable under the condition $\Delta t \le \Delta/\sqrt{2gH}$, where $\Delta = \max(\Delta x, \Delta y)$, $H = \max_{i,j}(H_{ij})$. The arrival of the wave to the coast is not considered in this work; it is assumed to arrive along the normal vector.

The boundary conditions on the bounds II–IV are replaced by their finite difference analogs.

The algorithm of solution of the direct problem is the following:

a) the initial data are defined: these are the distribution of depths in the domain Φ, the elevation of the bottom in the perturbation domain, the height of the wave and the initial velocity values;

b) the values of W_{n+1} are calculated from the known values of U_n, V_n, W_n;

c) the values of the velocities at the $(n+1)$-th time step are calculated from W_{n+1}, V_n, U_n.

Let us write down system (4) in the vector form:

$$A\vec{a} = \vec{b}. \tag{10}$$

Here the column vector $\vec{a}$ consists of the unknown coefficients a_{mn}:

$$\vec{a} = (a_{11}, \dots, a_{1n_y}, a_{21}, \dots a_{2n_y}, \dots, a_{n_x1}, \dots, a_{n_xn_y})^T.$$

Its length, i.e., the number of the unknown coefficients, is equal to $N = n_x \times n_y$.

The column vector $\vec{b}$ contains the right-hand sides of the equations:

$$\vec{b} = (w_{11}, \dots, w_{1n_l}, \dots, w_{p1}, \dots, w_{pn_l}, \dots, w_{k_p1}, \dots w_{k_pn_l})^T,$$

where p is a receiver number, n_l is the number of the time counts in each receiver.

The matrix A of system (10) is a rectangular one for a general case. The right singular vectors of the matrix A, i.e., $\{\vec{v}_i;\ i = 1, \dots, N\}$ make a basis in the space of solutions. In the same way, the left singular vectors $\{\vec{u}_j;\ j = 1, \dots M\}$ make a basis in the space of the right-hand sides. So, the solution to equation (10) is of the form:

$$\vec{a} = \sum_{j=1}^{N} \frac{(\vec{b} \cdot \vec{u_j})}{s_j} \vec{v}_j, \tag{11}$$

where s_j are singular values of the matrix A. In the process under study, the singular values rapidly decrease with an increase in their numbers, which is equivalent to the ill posedness of the problem. The approximated solution is therefore sought for as a linear combination of the first r right singular vectors:

$$\vec{a}_{[r]} = \sum_{j=1}^{\kappa} \frac{(\vec{b} \cdot \vec{u_j})}{s_j} \vec{v}_j. \tag{12}$$

The above-determined vector represents r-solution to system (10).

4. Numerical experiments

A series of calculations was made by the proposed method and were aimed at recovering the unknown function $\varphi(x,y)$. In all these calculations, the conditioning number of r-solution equals to the value cond $= 10^{-1}$.

The function to be recover was chosen in the form: $\varphi(x,y) = \psi(x,y)\alpha(x)$, where $\alpha(x)$ depends on the type of a model:

$$\alpha = \begin{cases} (x - x_0 + 3R_1)(x - x_0 + R_1/6) & \text{for Model 1,} \\ 1 & \text{for Model 2,} \end{cases}$$

and the function $\psi(x,y)$ describes the paraboloid:

$$\psi(x,y) = \max\left\{0,\ 1 - \frac{(x-x_0)^2}{{R_1}^2} - \frac{(y-y_0)^2}{{R_2}^2}\right\} \tag{13}$$

The domain Ω is $[100, 200] \times [50, 150]$ rectangle, the center point of the tsunami source $(x_0, y_0) = (150, 100)$ (all the sizes are assumed to be measured in kilometers). Function (13) was approximately sought for in the form of (9), where $n_x = 25$, $n_y = 11$. The receivers were disposed on the y-axis (the coastal line), the minimum distance between them was equal to 10 km, the receiver points belong to the segment $[10, 190]$, the maximum number of them is $k_p = 5$. The matrix A (as a real matrix) is of size 153×275 when there are three receivers, and 255×275 when there are five ones. By the word 'Model' we mean all information concerning the considered experiment.

The observed data concerning the form of the arrived wave were simulated as a result of solution of the direct problem (4)–(7), perturbed by the background noise, i.e., a high-frequency disturbance. All experiments presented here were made with the disturbance rate of 5% of a maximum amplitude of a signal over all receivers.

It is necessary to recognize that the results obtained strongly depend on the presence of disturbance due to the ill posedness of the problem. However, since a tsunami wave was much more lower-frequency as compared the background noise, is reasonable enough to apply the frequency filtration of the observed signal. In this work, the filtration is done by the method described in [9].

Model 1 is a source with the parameters: $R_1 = 25$, $R_2 = 50$, $\alpha(x) = (x - x_0 + 3R_1)(x - x_0 + R_1/6)$. Therefore, $f_{\max} \approx 0.73$. This model is the source of a gently dipping fault with incidence under an insular arc.

Model 2 differs from Model 1 by the function $\alpha(x)$: now $\alpha(x) = 1$; so, $f_{\max} = 1$. This source is a homogenous elevation near to circular paraboloid. Comparing the results obtained for these two sources we can understand clearly which features of coastal mareograms depend on a source and which features are due to a wave propagation.

Figures 1 and 2 represents the relative errors (in L_2-norm) of recovering the source functions for Models 1 and 2 from three and five receivers. Namely, Figures 1a and 1b correspond to these two models, respectively, for the case, when there are three receivers positioned symmetrically with respect to the midpoint of the coast. In each experiment, the horizontal axis points ($n = 1, \dots, 9$) note $n \times 10$ km the distance from the midpoint to the first supplemental pair of the receivers. By the midpoint we mean the projection of the central point of the rectangle Ω (or the central point of the source, that is the same) onto the coastal line. The presence of a receiver in the midpoint has an essential influence on the results because the signal in this direction is mostly informative. Figures 2a and 2b represent the same errors for Models 1 and 2, respectively, when five receivers are used.

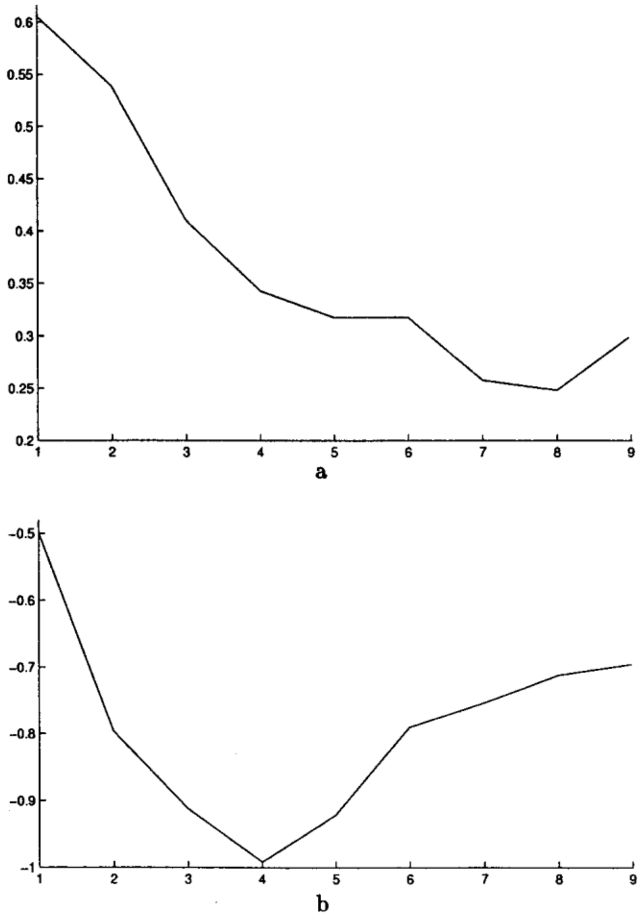
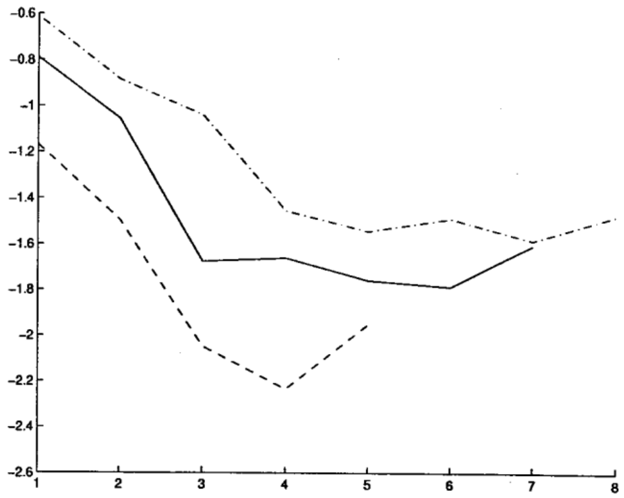
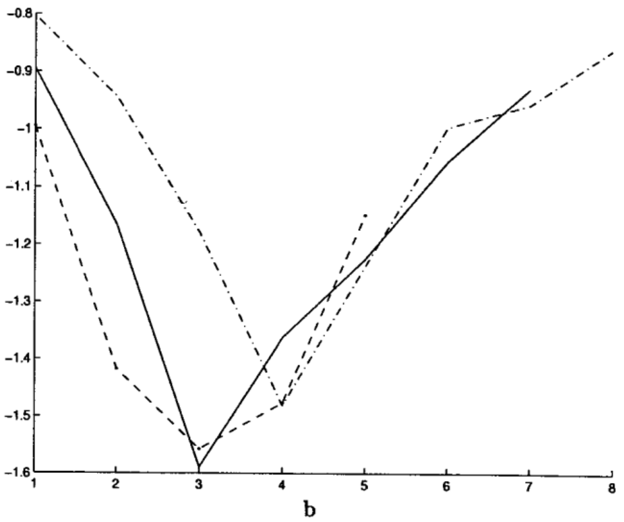


Figure 1. The relative errors (in L_2-norm) of recovering a source functions for models with 3 receivers positioned symmetrically with respect to the midpoint of the coast: a – Model 1, b – Model 2



a



b

Figure 2. The relative errors (in L_2-norm) of recovering a source functions for models with 5 receivers: a – Model 1, b – Model 2. One of the receivers is disposed at the midpoint and two pairs of others move symmetrically to the endpoints of the coastal segment while the distance in each pair is fixed at 40 km (dashed line), 20 km (solid line), and 10 km (dash-dot line).

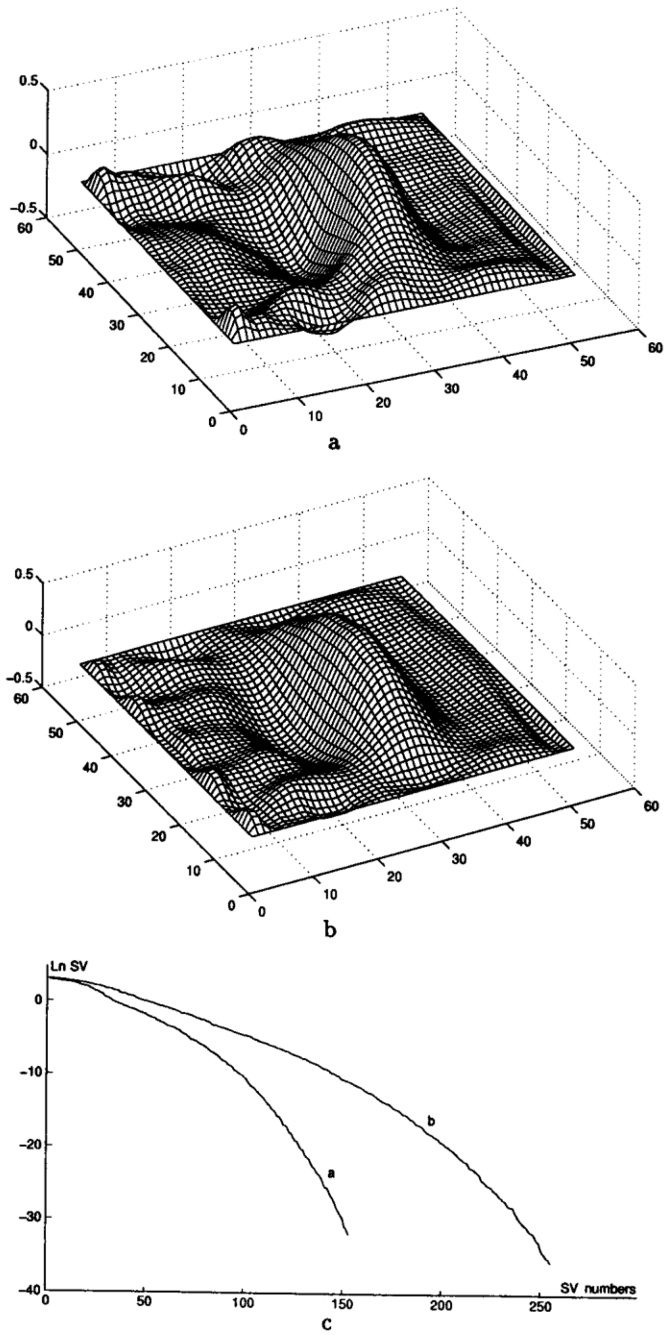


Figure 3. Calculations for Model 1. Recovered initial form (a) for 3 stations (err = 37.1%, app = 80 km, $f_{\max} = 0.63$ m, $f_{\min} = -0.282$ m) and (b) for 5 stations (err = 20.4%, app = 140 km, $f_{\max} = 0.71$ m, $f_{\min} = -0.29$ m). Graph (c) shows singular values (log scale) of the matrix A for the cases (a) and (b)

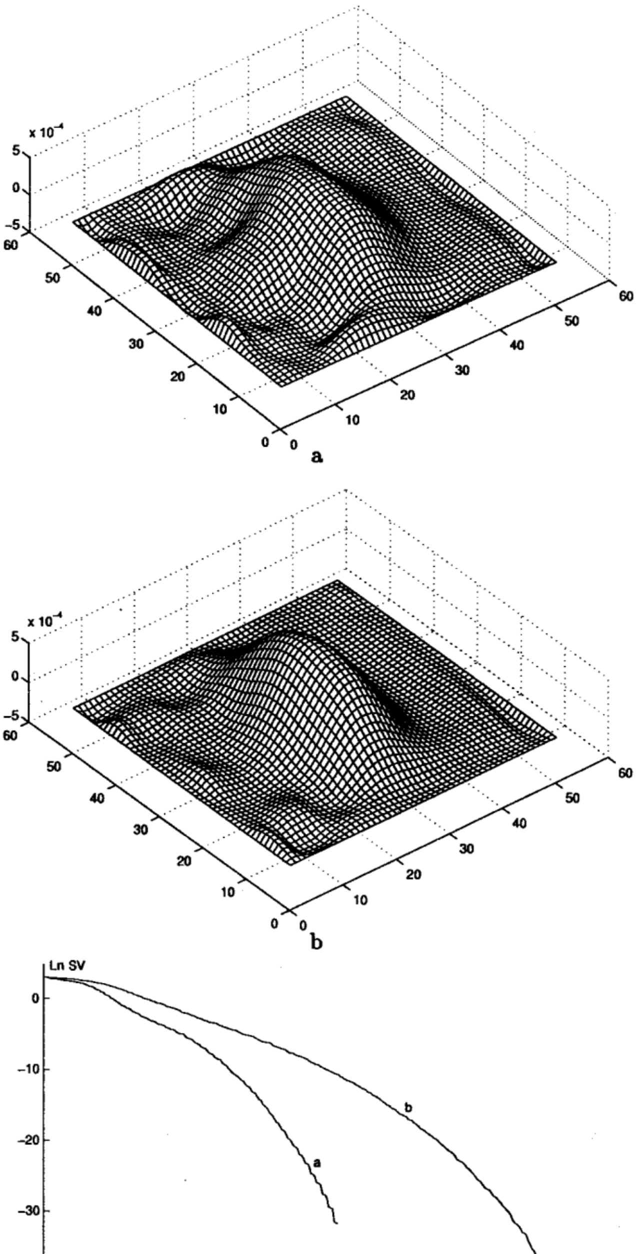


Figure 4. Calculations for Model 2. Recovered initial form (a) for 3 stations (err = 24.8%, app = 160 km, $f_{\max} = 0.87$ m; $f_{\min} = -0.15$ m) and (b) for 5 stations (err = 10.7%, app = 160 km, $f_{\max} = 0.99$ m, $f_{\min} = -0.09$ m). Graph (c) shows singular values (log scale) of the matrix A for the cases (a) and (b)

Figure 3 represents the natural logarithms of the singular values (Figure 3c) and source function recovered from three (Figure 3a) and five (Figure 3b) receivers for Model 1. Figure 4 contains the same results for Model 2. A sharp decrease in the singular values when their number increases is typical of all the calculations ('app' is a length of aperture).

The influence of the conditioning number is essential too. However, great oscillations appear in the solution when the decreasing the conditioning number (cond $< 10^{-1}$). This is typical of all ill-posed problems.

5. Conclusion

Based on conducted numerical experiments we can conclude:

1. The quality of reconstruction of the form of a source is improved with increasing the number of receivers, located within the characteristic size of the rectangle of the search and, also, in the case when the receiver captures the most informative signal;
2. Simulation of a source more complicated in its shape requires an increase in the number of basic functions and a decrease in the mesh size;
3. The application of R-solutions is an effective means of regularization of an ill-posed problem. The number of r basic vectors applied appears to be essentially lower than the minimum dimension of a matrix. This, in fact, enables us to avoid instability of the problem dealing with a sharp decrease of singular numbers of the matrix.

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