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Dispersion analysis of the hybrid plasma model*

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Abstract. This paper deals with the analysis of the hybrid plasma model based on the kinetic description of an ion component of the plasma and hydrodynamic approach for electrons. This type of models is widely used to investigate the processes and mechanisms of the collisionless interaction of interpenetrating plasma flows with a magnetic field with different values of the Mach numbers.

It is well known that nonstationary processes in the laboratory and the space plasma are always accompanied by the generation of different types of oscillations and waves. At the same time, the energy of a directed motion is transported to thermal and kinetic energies of the surrounding background, the electromagnetic field energy and fast particle flows. A general property of the phenomena of interaction of high-speed plasma flows is their collisionless nature with respect to Coulomb's collisions. This means that the flow interaction takes place at the distances which are essentially smaller than classical free paths. Under the current conditions, it is necessary to use the kinetic approach to describe the plasma behavior. Difficulties in the numerical implementation of the kinetic models, dealing with an essential difference in spatial-temporal scales for ions and electrons, resulted in developing the hybrid models. These models, based on the kinetic-hydrodynamic approach, are widely used in the research into a number of phenomena occurring in the laboratory and space plasma [1–4].

In this paper, the dispersion analysis of the hybrid plasma model in collisionless plasma is given. It allows finding a stability criterion for its numerical implementation.

The initial system of the hybrid model equations includes the kinetic Vlasov equation for ions:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{\boldsymbol{F}}{m_i} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0, \tag{1}$$

where $\boldsymbol{F} = e\Big(\boldsymbol{E} + \frac{1}{c}[\boldsymbol{v} \times \boldsymbol{B}]\Big)$, the equation of motion and the equation of an internal energy change for electrons are as follows:

$$m_e\left(\frac{\partial \boldsymbol{u}_e}{\partial t} + (\boldsymbol{u}_e \cdot \nabla)\boldsymbol{u}_e\right) = -e\left(\boldsymbol{E} + \frac{1}{c}[\boldsymbol{u}_e \times \boldsymbol{B}]\right) - \frac{\nabla p_e}{n}, \tag{2}$$

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$$n\Big(\frac{\partial T_e}{\partial t} + (\boldsymbol{u}_e \cdot \nabla) T_e\Big) = -(\gamma - 1) p_e \nabla \cdot \boldsymbol{u}_e; \tag{3}$$

and the Maxwell equations are

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j}, \qquad \nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}. \tag{4}$$

Here $f = f(\boldsymbol{r}, \boldsymbol{v}, t)$ is an ion velocity distribution function, m_i and e are a mass and a charge of ions, respectively, $\boldsymbol{v}$ is an ion velocity, $\boldsymbol{B}$ and $\boldsymbol{E}$ are the intensities of magnetic and electric fields, m_e and $\boldsymbol{u}$ are the mass and velocity of electrons, $p_e = nT_e$ is a pressure of an electron component, T_e is an electron temperature, $\boldsymbol{j} = en(\boldsymbol{V}_i - \boldsymbol{v}_e)$ is the current density, and γ is the ratio of specific heats. The plasma is quasi-neutral, i.e., it is assumed that $n_e = n_i = n$. The ion density and the average ion velocity are found as moments of the ion distribution function

$$n = n_i = \int f(\boldsymbol{r}, \boldsymbol{v}, t)\, d\boldsymbol{v}, \qquad \langle \boldsymbol{v} \rangle = \boldsymbol{V}_i = \frac{1}{n}\int \boldsymbol{v} f(\boldsymbol{r}, \boldsymbol{v}, t)\, d\boldsymbol{v}.$$

The numerical approach to solving system of equations (1)–(4) includes the particle-in-cell (PIC) method and finite difference schemes [3–5]. A more detailed description of the algorithm to solve this system is presented in [6]. In this paper, we investigate the dispersion properties of the above hybrid model. The dispersion relation for plasma in a general case is derived in [7].

Let us turn from the Vlasov equation to the equations for moments with allowance for the ion dispersion by velocities (i.e., the ion temperature) to be equal to zero. Consequently, we obtain only two equations, instead of an infinite system of equations for moments

$$m_i\Big(\frac{\partial \boldsymbol{v}_i}{\partial t} + (\boldsymbol{v}_i \cdot \nabla)\boldsymbol{v}_i\Big) = e\Big(\boldsymbol{E} + \frac{1}{c}(\boldsymbol{v}_i \times \boldsymbol{B})\Big), \tag{5}$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \boldsymbol{v}_i) = 0. \tag{6}$$

The electrons are considered in the approach of a massless fluid, i.e., the electron inertia is omitted.

Let us write down the system of equations (2)–(6) in the Cartesian coordinates when considering the problem of the the wave oscillation propagation along the axis x $\Big(\frac{\partial}{\partial y} = 0$ and $\frac{\partial}{\partial z} = 0\Big)$ which, initially ($t = 0$), takes place in the uniform and equilibrium plasma:

$$m_i\Big(\frac{\partial v_{ix}}{\partial t} + v_{ix}\frac{\partial v_{ix}}{\partial x}\Big) = eE_x + \frac{e}{c}v_{iy}B_z - \frac{e}{c}v_{iz}B_y, \tag{7}$$

$$m_i\Big(\frac{\partial v_{iy}}{\partial t} + v_{ix}\frac{\partial v_{iy}}{\partial x}\Big) = eE_y + \frac{e}{c}v_{iz}B_x - \frac{e}{c}v_{ix}B_z, \tag{8}$$

$$m_i\Big(\frac{\partial v_{iz}}{\partial t}+v_{ix}\frac{\partial v_{iz}}{\partial x}\Big)=eE_z+\frac{e}{c}v_{ix}B_y-\frac{e}{c}v_{iy}B_x, \tag{9}$$

$$\frac{\partial n_i}{\partial t}+n_i\frac{\partial v_{ix}}{\partial x}=0, \tag{10}$$

$$eE_x+\frac{e}{c}v_{ey}B_z-\frac{e}{c}v_{ez}B_y+\frac{1}{n_e}\frac{\partial p_e}{\partial x}=0, \tag{11}$$

$$eE_y+\frac{e}{c}v_{ez}B_x-\frac{e}{c}v_{ex}B_z=0, \tag{12}$$

$$eE_z+\frac{e}{c}v_{ex}B_y-\frac{e}{c}v_{ey}B_x=0, \tag{13}$$

$$n_e\Big(\frac{\partial T_e}{\partial t}+v_{ex}\frac{\partial T_e}{\partial x}\Big)+p_e(\gamma-1)\frac{\partial v_{ex}}{\partial x}=0, \tag{14}$$

$$n_iv_{ix}-n_ev_{ex}=0, \tag{15}$$

$$\frac{\partial B_z}{\partial x}=-\frac{4\pi e}{c}(n_iv_{iy}-n_ev_{ey}), \tag{16}$$

$$\frac{\partial B_y}{\partial x}=\frac{4\pi e}{c}(n_iv_{iz}-n_ev_{ez}), \tag{17}$$

$$\frac{\partial B_x}{\partial t}=0, \tag{18}$$

$$\frac{\partial E_z}{\partial x}=\frac{1}{c}\frac{\partial B_y}{\partial t},\qquad \frac{\partial E_y}{\partial x}=-\frac{1}{c}\frac{\partial B_z}{\partial t}. \tag{19}$$

Then let us assume $f=f^0+f^*$, where f is the following function $f=f\{v_{ix},v_{iy},v_{iz},n,v_{ex},v_{ey},v_{ez},T,B_x,B_y,B_z,E_x,E_y,E_z\}$. Here f^0 is the value of the function in the initial non-perturbed state and f^* is the initial perturbation ($f^*\ll f^0$). For the case in question, $\boldsymbol{v}_0=0$, $\boldsymbol{B}_0=(B_x^0,\,B_y^0,\,0)=(B^0\cos\alpha,B^0\sin\alpha,0)$, where B^0 is the amplitude of a non-perturbed field and α is the angle between the directions of the magnetic field and the axis x. Linearizing system of equations (7)–(19) assuming $f^*=\tilde f\exp(-i\omega t+ik_\parallel x)$, where ω is the frequency of oscillations and $k_\parallel$ is a wave number, we obtain the following cubic equation for $x=\Big(\frac{\omega}{k_\parallel}\Big)^2$:

$$\begin{aligned}&(4\pi m_in_0)^2x^3-\\&\quad\Big[\Big(\frac{1}{e}m_ik_\parallel cB_x^0\Big)^2+4\pi m_in_0(2(B_x^0)^2+(B_y^0)^2)+(4\pi n_0)^2m_i\gamma T_0\Big]x^2+\\&\qquad(B_x^0)^2\Big[(B_x^0)^2+(B_y^0)^2+\frac{1}{e^2}m_i(k_\parallel c)^2\gamma T_0+8\pi n_0\gamma T_0\Big]x-\\&\qquad\quad\frac{1}{m_i}\gamma T_0(B_x^0)^4=0.\end{aligned} \tag{20}$$

From equation (20) one can see that the signs of the coefficients do not change when varying parameters. Consequently, at $x=0$ the function signs and derivatives alternate: $f(0) < 0$, $f'(0) > 0$, $f''(0) < 0$, and $f'''(0) > 0$, that is, there are three sign inversions. On the other hand, with sufficiently large $x = x^*$, the sign of the function and its derivatives is defined by the first term of the expression, i.e., by the sign of the coefficient a_0 ($f = a_0x^3 + a_1x^2 + a_2x + a_3$). Then there are no sign inversions in the sequence of the function values and its derivatives at x^*. Consequently, according to the Sturm's theorem, there are exactly three roots on the interval $x = [0, x^*]$. As $(\omega/k_\|)^2 > 0$, then ω is a real number and, hence, f^* has the oscillating behavior with respect to time. This results in that the system of equations is stable. These roots allow one to define the wave phase velocity $v_{ph} = \omega/k_\|$.

Dividing equation (20) by $(4\pi m_i n_0)^2$ and taking into account the fact that the Alfven velocity is $v_A = \dfrac{B^0}{\sqrt{4\pi n_i m_i}}$, the ion cyclotron frequency is $\omega_{ci} = \dfrac{eB^0}{m_i c}$, the ion plasma frequency is $\omega_{pi} = \sqrt{\dfrac{4\pi n_i e^2}{m_i}}$ and the sound speed is $c_s = \sqrt{\dfrac{\gamma T_e}{m_i}}$, we arrive at the following equation:

$$x^3 - v_A^2\left(\frac{k_\|^2c^2}{\omega_{pi}^2}\cos^2\alpha + 1 + \cos^2\alpha + \frac{c_s^2}{v_A^2}\right)x^2 + v_A^2\cos^2\alpha\left(v_A^2 + \frac{k_\|^2c^2}{\omega_{pi}^2}c_s^2 + 2c_s^2\right)x - c_s^2v_A^4\cos^4\alpha = 0 \tag{21}$$

where $x = \left(\dfrac{\omega}{k_\|}\right)^2$. Next, let us proceed to dimensionless variables choosing as characteristic parameters the velocity v_A and the wave number ω_{pi}/c. In dimensionless variables, equation (21) is as follows:

$$x^3-\left(k_\|^2\cos^2\alpha+1+\cos^2\alpha+c_s^2\right)x^2+\cos^2\alpha\left(1+k_\|^2c_s^2+2c_s^2\right)x-c_s^2\cos^4\alpha = 0. \tag{22}$$

The contours of the phase velocity for the value of $c_s = 0.5$ are given in Figure 1. The angle $\alpha = 0$ refers to the maximal values of v_{ph}. Figure 2 illustrates the dependence of the phase velocity v_{ph} on the wave number $k_\|$, obtained when solving equation (22) for $c_s = 0.5$ and $\alpha = 0$. The first minor roots show that $v_{ph}(k_\|)$ does not grow when $k_\|$ increases, and the third (maximal) root grows. Also, at large values of $k_\|$ the phase velocity is proportional to $k_\|$, $v_{ph} \sim k_\|$, i.e., the smaller the wave length of the oscillation, the larger its velocity. Thus, the velocity of the perturbation propagation can turn out to be infinitely large, even if its amplitude is very small.

When using the numerical methods for solving problems based on the considered model, the phase velocity cannot be infinite. In any numerical grid method, there is a minimal size equal to a minimal grid step. Consequently, there is a minimal wave length and, respectively, the maximal wave

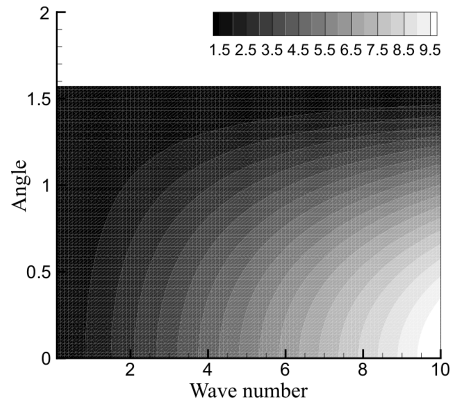

Figure 1. Contours of the phase velocity v_{ph}/v_A depending on the wave number $k_{\|}/k_0$ and the angle α when $c_s = 0.5$

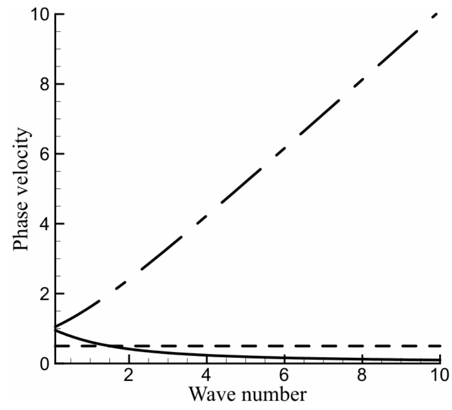

Figure 2. Phase velocity v_{ph}/v_A depending on the wave number $k_{\|}/k_0$. Here the sound speed is $c_s = 0.5$ and $\alpha = 0$

number $k_{\|,\max} = 2\pi/h$, where h is a minimal grid step. Then the maximal phase velocity $v_{ph,\max}$ is limited. The well-known stability criterion (the Courant condition) combining time and space steps is $\frac{\tau v_{ph}}{h} \leq 1$, and in the case of the considered model it becomes

$$\frac{2\pi\tau}{h^2} \leq 1.$$

From this relation, it follows that τ has to be chosen proportional to h^2. The hybrid numerical model of plasma is explicit and, consequently, when implementing it, it is necessary to take into account this stability condition which differs from the one for the hyperbolic systems.

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