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Coordinate transformation for a small-scale meteorological model*

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A terrain-following coordinate transformation is introduced into a small-scale meteorological model by using a variational approach based on the Hamiltonian principle. The coordinate transformation is explicitly formulated in the corresponding Lagrange function. In this way, limitations on orography steepness can be easily obtained. An example of a mountain wave calculation with the small-scale model is given.

1. Introduction

Terrain-following coordinate transformations are widely used in computational meteorology [1]. They are most popular tools to calculate problems with smooth terrain orography. However, these transformations have some limitations on orography steepness. For instance, in a survey paper on non-hydrostatic meteorological models [2], each of the many models considered has its limitation. It is not always an easy task to estimate theoretically the maximum permissible orography steepness for a model. Such estimates are usually obtained by numerical experimentation.

In this paper, a coordinate transformation is used in a small-scale meteorological model by a variational approach based on the Hamiltonian principle. The coordinate transformation is explicitly introduced into the corresponding Lagrange function. Some steepness limitations can be obtained in this way.

The paper is organized as follows: In Section 2, the variational approach to coordinate transformations is described. Section 3 is devoted to a small-scale meteorological model based on a coordinate transformation of the type discussed above. To conclude, an example of a smooth orography calculation with the model is given as an illustration in Section 4.

2. Variational formulation

Consider the following primitive equations of motion:

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$$\frac{du}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial x}=fv, \qquad \frac{dv}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial y}=-fu, \qquad \frac{dw}{dt}=-g.$$

In this section, the equation of continuity and thermodynamic equation are not considered. The molecular friction terms are assumed to be neglected here. Let x, y, and z be Cartesian coordinates, and $h(x,y)$ the orography height. H is assumed to be constant.

Under coordinate transformation $\eta = \frac{z-h}{H-h}$ we have the new and old coordinates as follows:

$$\begin{aligned} &x^1 = x, && x = x^1, \\ &x^2 = y, && y = x^2, \\ &x^3 = \eta(x,y,z), && z = \xi(x^1,x^2,x^3) \quad \text{at } \xi = \eta(H-h)+h. \end{aligned}$$

Under the transformation, the original complex domain becomes a rectangle. Both systems of coordinates are related by conventional formulas from the theory of metric spaces.

For covariant vectors, we have from $q^i = \nabla x^i$ that

$$q^1 = i, \qquad q^2 = j, \qquad q^3 = \frac{\eta-1}{H-h}\frac{\partial h}{\partial x}i + \frac{\eta-1}{H-h}\frac{\partial h}{\partial y}j + \frac{1}{H-h}k.$$

And for covariant vectors, from $q_i = \frac{\partial x}{\partial x^i}$:

$$q_1 = i + (1-\eta)\frac{\partial h}{\partial x}k, \qquad q_2 = j + (1-\eta)\frac{\partial h}{\partial y}k, \qquad q_3 = (H-h)k.$$

For the metric tensor we have in contravariant form

$$q^{ij} = q^i q^j = \begin{pmatrix} 1 & 0 & \frac{\eta-1}{H-h}\frac{\partial h}{\partial x} \\ 0 & 1 & \frac{\eta-1}{H-h}\frac{\partial h}{\partial y} \\ \frac{\eta-1}{H-h}\frac{\partial h}{\partial x} & \frac{\eta-1}{H-h}\frac{\partial h}{\partial y} & q^{ij}_{33} \end{pmatrix},$$

$$q^{ij}_{33} = \left(\frac{1}{H-h}\right)^2 + \left(\frac{\eta-1}{H-h}\right)^2 \left\{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right\},$$

and in covariant form

$$q_{ij} = q_i q_j = \begin{pmatrix} 1+(1-\eta)^2\left(\frac{\partial h}{\partial x}\right)^2 & (1-\eta)^2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y} & (1-\eta)(H-h)\frac{\partial h}{\partial x} \\ (1-\eta)^2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y} & 1+(1-\eta)^2\left(\frac{\partial h}{\partial y}\right)^2 & (1-\eta)(H-h)\frac{\partial h}{\partial y} \\ (1-\eta)(H-h)\frac{\partial h}{\partial h} & (1-\eta)(H-h)\frac{\partial h}{\partial y} & (H-h)^2 \end{pmatrix}.$$

The following relation between the new and old velocity components is valid:

$$u^1 = u, \qquad u^2 = v, \qquad u^3 = \frac{\eta - 1}{H - h} u \frac{\partial h}{\partial h} + \frac{\eta - 1}{H - h} v \frac{\partial h}{\partial y} + \frac{1}{H - h} w = \omega.$$

Now, we reformulate the equations of motions into Hamiltonian form. For this, we construct the Lagrange function $L = T - U$. From the above formulas we have for the kinetic energy per unit mass in an absolute, non-rotational system of coordinates:

$$T = \frac{1}{2}(u^2 q_{11} + v^2 q_{22} + \omega^2 q_{33}) + uvq_{12} + u\omega q_{13} + v\omega q_{23}.$$

From this, the effect of orography steepness in the equations of motion can be estimated as follows. We assume that $\nabla h \ll 1$, that is, the orography is smooth enough. Thus, the terms outside the brackets become small and can be neglected. The order of smallness can be estimated according to assumptions on orography steepness. Returning to the rotation coordinate system, we have

$$T = \frac{1}{2}(u^2 + v^2 + \omega^2 (H - h)^2) - \frac{f}{2}(yu - xv) + \frac{f^2}{8}(x^2 + y^2).$$

The potential energy U is the sum of potentials of the gravity and the rotational force:

$$U = g(\eta(H - h) + h) + \frac{f^2}{8}(x^2 + y^2).$$

The Lagrange function is thus defined.

From the Hamiltonian principle, we have

$$\frac{d}{dt}\left(\frac{\partial (T - U)}{\partial \dot{r}}\right) - \frac{\partial (T - U)}{\partial r} = 0,$$

where $\dot{r} = (u, v, \omega)$ and $r = (x, y, \eta)$.

The Euler–Lagrange equations can be obtained now as follows:

$$\frac{du}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial h}{\partial x}(g(1 - \eta) + (H - h)\omega^2) = fv,$$

$$\frac{dv}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial h}{\partial y}(g(1 - \eta) + (H - h)\omega^2) = -fu,$$

$$\frac{d\omega}{dt} + \frac{1}{(H - h)^2}\frac{1}{\rho}\frac{\partial p}{\partial \eta} - \frac{2\omega}{H - h}\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) = \frac{-g}{H - h}.$$

These equations can be further simplified by the Boussinesq approximation and splitting of the meteorological variables into a basic state and deviations [1]. In the next section, we consider a model reduced to such a form.

3. The model

From the computational viewpoint, it is possible, but not effective, to use the above formulas in numerical calculations.

A much better way is to calculate the third equation of motion for the old vertical velocity component rather than for the new one [3]. This is done in the following meteorological model.

The basic equations of motion, heat, moisture, and continuity in a terrain-following coordinate system are as follows:

$$\frac{dU}{dt} + \frac{\partial P}{\partial x} + \frac{\partial(G^{13}P)}{\partial \eta} = f_1(V - V_g) - f_2W + R_u,$$

$$\frac{dV}{dt} + \frac{\partial P}{\partial y} + \frac{\partial(G^{23}P)}{\partial \eta} = -f_1(U - U_g) + R_v,$$

$$\frac{dW}{dt} + \frac{1}{G^{1/2}}\frac{\partial P}{\partial \eta} + \frac{gP}{C_s^2} = f_2U + g\frac{G^{1/2}\bar{\rho}\theta'}{\bar{\theta}} + R_w$$

$$\frac{d\theta}{dt} = R_\theta,$$

$$\frac{ds}{dt} = R_s,$$

$$\frac{1}{C_s^2}\frac{\partial P}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial}{\partial \eta}\Big(G^{13}U + G^{23}V + \frac{1}{G^{1/2}}W\Big) = \frac{\partial}{\partial t}\Big(\frac{G^{1/2}\bar{\rho}\theta'}{\bar{\theta}}\Big).$$

Here $U = \bar{\rho}G^{1/2}u$, $V = \bar{\rho}G^{1/2}v$, $W = \bar{\rho}G^{1/2}w$, $P = G^{1/2}p'$, where p' and θ' are deviations from the basic state pressure $\bar{p}$ and potential temperature $\bar{\theta}$, s is the specific humidity, C_s is the sound wave speed, u_g, v_g are the components of geostrophic wind representing the synoptic part of the pressure, $\eta = \frac{H(z-z_s)}{(H-z_s)}$ is a terrain-following coordinate transformation, z_s is the surface height, H is the height of the top of the model domain. In the model, $H = \text{const}$,

$$G^{1/2} = 1 - \frac{z_s}{H}, \qquad G^{13} = \frac{1}{G^{1/2}}\Big(\frac{\eta}{H} - 1\Big)\frac{\partial z_s}{\partial x}, \qquad G^{23} = \frac{1}{G^{1/2}}\Big(\frac{\eta}{H} - 1\Big)\frac{\partial z_s}{\partial y}.$$

In the above equations we use the following notation: for an arbitrary function φ

$$\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t} + \frac{\partial u\varphi}{\partial x} + \frac{\partial v\varphi}{\partial y} + \frac{\partial \omega\varphi}{\partial \eta} = \frac{\partial \varphi}{\partial t} + \mathrm{ADV}\varphi,$$

where

$$\omega = \frac{1}{G^{1/2}}w + G^{13}u + G^{23}v.$$

The terms R_u, R_v, R_ω, R_θ, R_s refer to subgrid-scale processes. As the turbulence parameterization of the model, we use a simple scheme

$$K_m = \begin{cases} l^2\sqrt{\frac{1}{2}\underline{D}^2(1-\mathrm{Ri})}, & \mathrm{Ri} < 1, \\ 0, & \mathrm{Ri} \geq 1, \end{cases}$$

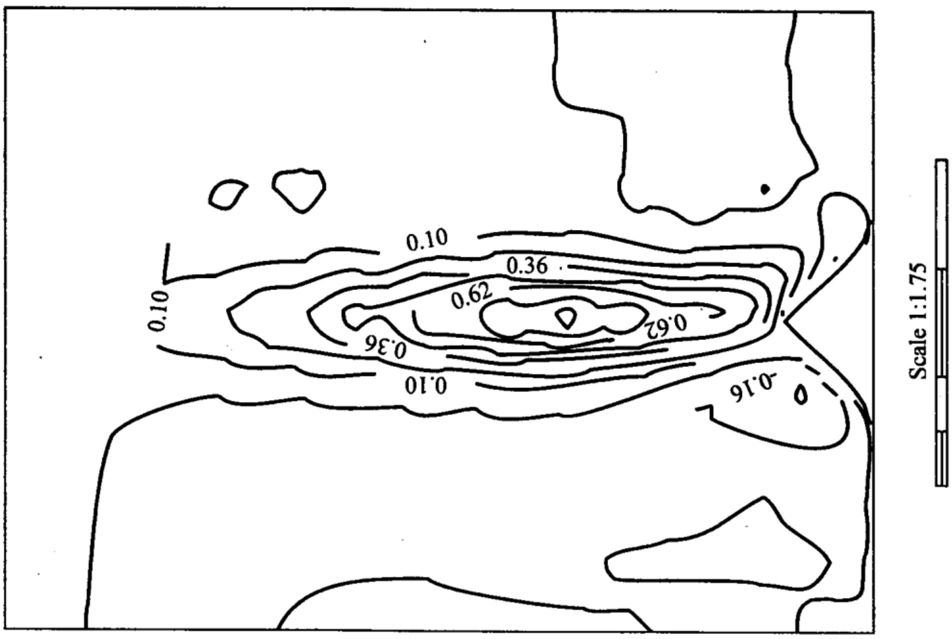
$$\mathrm{Ri} = \frac{g(d\ln\theta/dz)}{D^2/2}, \quad D = \nabla\underline{u} + \underline{u}\nabla,$$

where K_m is the vertical exchange coefficient, Ri is the local Richardson number, l is the Blackadar mixing length [1]. A more detailed description of the model and the numerical methods used for calculation, can be found in [4, 5].

4. Example

In this section, an example of mountain wave calculation over smooth orography is given.

A bell-shaped mountain with a height of 500 m is located at the center of a 10 × 10 km domain. The top of the domain is at 5 km. A geostrophic flow goes from the west, with $u_g = 5$ m/s and $v_g = 0$. As the basic state, a standard atmospheric stratification $\frac{d\theta}{dz} = 3.5$ K/km is assumed. An absorbing layer is located above a height of approximately 1500 m. The com-



putational grid consists of $31 \times 31 \times 16$ points, the horizontal grid size is $\Delta x = \Delta y = 333$ m, the vertical grid size Δz is variable, increasing with height. The mountain is slowly inflated during the first 15 minutes of the computation.

A steady-state is achieved after one hour of physical time. The figure shows the vertical velocity component at a north-south cross-section over the center of the mountain. The mountain here is smooth and not steep, therefore the above coordinate transformation is valid.

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