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Group classification of a one-dimensional nonlinear poroelasticity equation

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Abstract. This paper studies a class of second order partial differential equations $u_{tt} = f(u_x)u_{xx} + g(u_t)$ arising in poroelasticity theory with arbitrary functions $f(u_x)$ and $g(u_t)$, using the group classification. It is shown that the principal Lie algebra of infinitely infinitesimal symmetries is three-dimensional. We use the method of a preliminary group classification for obtaining the classifications of these equations for a one-dimensional extension of the main Lie algebra.

Keywords: porous medium, permeability, nonlinear equation, displacement rate, infinitesimal operator, Lie algebra.

1. Introduction

The presence of water and gas in underground reservoirs leads to phase shifts and dependence on the frequency of changes in the amplitude of seismic waves (for example, [1,2]). In [3–13], a two-phase model of the medium was introduced to describe the interrelated wave propagation in a porous fluid-saturated medium. Much attention is also given to the dissipation models of a porous medium saturated with liquid and the ways of their account in the equations of state.

The fundamental property of an elastically deformed saturated liquid of a porous medium, which follows from the theory of poroelasticity, is that in such media, two longitudinal waves can propagate: fast and slow, as well as a transverse wave.

In [14], a diffusion approach was proposed for a one-dimensional homogeneous model of porous media. Compared to [14], important improvements have been made: a good representation of viscous dissipation in the entire frequency range; model optimization; evaluation of computational efforts in terms of the required accuracy.

In [15], a closed one-dimensional system of first order dynamic integro-differential equations was obtained for the velocity components of the vector displacements of an elastic porous body, saturating fluid, and stress tensor in the dissipative approximation due to the coefficient of permeability. The proposed mathematical model is thermodynamically consistent and satisfies the first physical principles. The dependence of the dispersion ratio of the resulting system on the physical and kinetic parameters is revealed.

2. One-dimensional dynamic poroelasticity equation for transverse waves in the dissipative approximation

Let us consider the propagation of nonlinear transverse seismic waves in the case when for the sake of simplicity the partial density of the porous matrix assume of to be equal to unity, and the shear modulus μ is a function of the strain rate, and the friction force, which determines the energy dissipation is a function only of the phase velocity. Under such assumptions, the nonlinear one-dimensional equation of poroelasticity can be written down in the following form [16, 17]:

$$u_{tt} = f(u_x)u_{xx} + g(u_t), \tag{1}$$

where u is the velocity of the porous matrix, $u_t = \dfrac{\partial u}{\partial t}$ and $u_x = \dfrac{\partial u}{\partial x}$ are the differentiation operators.

An equation of the form (1) arises when studying a one-dimensional dynamic system of equations for the poroelasticity of SH-waves in the case of low porosity and low permeability [16, 18]. The function g is related to the permeability of a porous medium and is responsible for the dissipation energy, while the function f is responsible for the velocity propagation of the transverse wave. The velocity of the saturating fluid v is explicitly defined in terms of the functions u [17].

3. Invariant transformations and the Lie algebra principle

Following [19], for equation (1), we write down the invariance condition

$$\underset{2}{X}\big[u_{tt} = f(u_x)u_{xx} + g(u_t)\big]\Big|_{(1)} = 0, \tag{2}$$

where for the second prolongation $\underset{2}{X}$ of the infinitesimal operator

$$X = \xi_1(t,x,u)\frac{\partial}{\partial t} + \xi_2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{3}$$

the following prolongation formulas are obtained

$$\underset{2}{X} = X + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{11}\frac{\partial}{\partial u_{tt}} + \zeta_{22}\frac{\partial}{\partial u_{xx}} \tag{4}$$

and the coordinates of the prolongated operator ζ_1, ζ_2, ζ_{11}, ζ_{22} in the extended space are defined by the well-known formulas [20, 21].

Substituting (3) and (4) into (2), we obtain the following determining equation:

$$\big[\zeta_{11} - f\zeta_{22} - u_{xx}\zeta_2 f_{u_x} - \zeta_1 g_{u_t}\big]\Big|_{(1)} = 0.$$

Hence, due to the arbitrariness of the functions f and g, we obtain

$$\zeta_1 = \zeta_2 = \zeta_{11} = \zeta_{22} = 0$$

or

$$(\xi_k)_t = (\xi_k)_x = (\xi_k)_u = 0, \quad k = 1, 2,$$
$$\eta_t = \eta_x = \eta_u = 0.$$

From these relationships, after simple transformations, we obtain

$$\xi_1 = c_1, \qquad \xi_2 = c_2, \qquad \eta = c_3.$$

Consequently, for the arbitrary functions f and g, equation (1) admits a three-dimensional Lie algebra L_3 with a basis

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial u}.$$

Following [19], we will call L_3 the main Lie algebra for equation (1). The remainder of the group classification should represent the coefficients f and g such that equation (1) admits an extension of the main algebra L_3. To this end, for convenience, we introduce the new notation $f^1 = f$ and $f^2 = g$.

An equivalence transformation is a non-degenerate change of the variables t, x, u taking any equation of the form (1) into an equation of the same form, generally speaking, with different $f(u_x)$ and $g(u_t)$. A set of all equivalences of the transformation forms an equivalence group E. Following [20], we must find a continuous subgroup E_c, using the infinitely infinitesimal method.

The group operator E_c takes the form

$$Y = \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \mu^1 \frac{\partial}{\partial f^1} + \mu^2 \frac{\partial}{\partial f^2} = X + \mu^1 \frac{\partial}{\partial f^1} + \mu^2 \frac{\partial}{\partial f^2},$$

where X is defined by formula (3). The dependencies of f^k and μ^k are as follows: $f^k = f^k(t, x, u, u_t, u_x)$ and $\mu^k = \mu^k(t, x, u, u_t, u_x, f^1, f^2)$. Equation (1) is written down as the following system:

$$u_{tt} - f^1(u_x) u_{xx} - f^2(u_t) = 0, \tag{5}$$

$$f^k_t = f^k_x = f^k_u = f^k_{u_t} = f^k_{u_x} = 0, \quad k = 1, 2. \tag{6}$$

The invariance conditions for this system are the following:

$$\underset{2}{Y} = Y + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{11} \frac{\partial}{\partial u_{tt}} + \zeta_{22} \frac{\partial}{\partial u_{xx}} + \\ \omega^k_2 \frac{\partial}{\partial f^k_x} + \omega^k_1 \frac{\partial}{\partial f^k_t} + \omega^k_0 \frac{\partial}{\partial f^k_0} + \omega^1_{01} \frac{\partial}{\partial f^k_{u_t}} + \omega^1_{10} \frac{\partial}{\partial f^k_{u_x}}. \tag{7}$$

Here, the coefficients ζ_1, ζ_2, ζ_{11}, ζ_{22} of operator (4) and other coefficients (7) are obtained by applying the procedure to the differential variables f^k with the independent variables (t, x, u, u_t, u_x).

After simple transformations, we obtain the following continuation formulas:

$$\omega_0^k = (\mu^k)_u - f^k_{u_t}(\xi_1)_u - f^k_{u_x}(\xi_2)_u,$$
$$\omega_1^k = (\mu^k)_t - f^k_{u_t}(\zeta_1)_t - f^k_{u_x}(\zeta_2)_t,$$
$$\omega_2^k = (\mu^k)_x - f^k_{u_t}(\zeta_1)_x - f^k_{u_x}(\zeta_2)_x,$$
$$\omega_{01}^1 = (\mu^1)_{u_t} - f^1_{u_x}\big((\zeta_2)_{u_t} + f^2_{u_t}(\zeta_2)_{f^2}\big),$$
$$\omega_{10}^1 = (\mu^1)_{u_x} - f^1_{u_x}\big((\zeta_2)_{u_x} + f^1_{u_x}(\zeta_2)_{f^1}\big).$$

Hence, taking into account the invariance condition (6), we obtain

$$\omega_j^k = 0, \quad k = 1, 2, \;\; j = 0, 1, 2, \qquad \omega_{01}^1 = 0, \quad \omega_{10}^1 = 0.$$

Since these relations must be satisfied for any f^1 and f^2, we obtain

$$(\zeta_1)_t = (\zeta_2)_t = (\zeta_1)_x = (\zeta_2)_x = 0,$$
$$(\zeta_1)_u = (\zeta_2)_u = (\zeta_1)_{u_x} = (\zeta_2)_{u_t} = 0,$$
$$(\mu^k)_t = (\mu^k)_x = (\mu^k)_u = 0,$$
$$(\mu^1)_{u_t} = (\mu^2)_{u_x} = 0.$$

After simple calculations we come to

$$\xi_1 = \xi_1(t), \quad \xi_2 = \xi_2(x), \quad \mu^1 = \mu^1(u_x, f^1, f^2), \quad \mu^2 = \mu^2(u_t, f^1, f^2),$$
$$\eta = c_1 u + c_2 x + c_3 t + c_4, \quad c_m = \text{const}, \quad m = 1, 2, 3, 4.$$

The remaining invariance condition for (1) with allowance for (7) can be written as

$$\zeta_{11} - f^1\zeta_{22} - \mu^1 u_{xx} - \mu^2 = 0.$$

Hence, taking into account equation (1), we obtain

$$-(\xi_1)'' u_t + \big[(2c_1 - 3(\xi_1)')f^1 - \mu^1 - (2c_1 - 3(\xi_2)')f^1\big]u_{xx} +$$
$$(2c_1 - 3(\xi_1)')f^2 + (\xi_2)'' u_x f^1 - \mu^2 = 0.$$

Since this is the ratio of u, u_t, u_x and u_{xx} are independent variables, we have

$$\eta = c_1 u + c_2 x + c_3 t + c_4, \quad \xi_1 = c_5 t + c_6, \quad \xi_2 = \psi(x),$$
$$\mu^1 = 3(\psi' - c_5)f^1, \quad \mu^2 = [2c_1 - 3c_5]f^2 + \psi'' u_x f^1,$$

where c_1, c_2, c_3, c_4, c_5, c_6 are arbitrary constants and ψ is an arbitrary function.

Conclusion

The class of second order partial differential equations $u_{tt} = f(u_x)\, u_{xx}+g(u_t)$ arising in the theory of poroelasticity is being studied. The arbitrary functions $f(u_x)$ and $g(u_t)$ correspond to the propagation velocity of shear waves and the dissipation of energy due to the permeability of the medium, respectively. It is shown that the Lie algebra of infinitely infinitesimal symmetries is three-dimensional. Also, we have carried out a group analysis of this class of equations based on the method of the preliminary group classification.

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