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The integral equation of the convolution type in inverse problems of the theory of wave propagation

A.S. Zapreev

A number of inverse problems of oscillation processes studied in optics, acoustics, radiophysics and geophysics may be reduced to solving the integral equation of the convolution type with the special kernel. Several theorems for uniqueness solution of this integral equation are proved. These proofs are constructive and may be used for calculation solution of such inverse problems.

1. Introduction

In the theory of wave propagation a current problem of great applied importance is that of determining the properties of a medium based on a wave field known on some manifold. A number of oscillation processes studied in optics, acoustics, radiophysics, and geophysics are described with sufficient accuracy by the wave equation. The problems in such conditions thus often lead to inverse problems for the wave equation that in a number cases reduce to inverse problems of determining the right-hand side of the Helmholtz equation. The latter inverse problem may be reduced to solving the equivalent problems for the integral equation.

The main results obtained in the investigation of the Helmholtz equation are connected with the inverse problem of metaharmonic potentials. The investigations in this direction were carried out by P.S. Novikov, L.N. Sretenskii, V.K. Ivanov, M.M. Lavrent'ev, A.I. Prilepko [1], V.G. Cherednitchenko [2], V.M. Isacov, V.G. Pavlov [3] and others. A coefficient inverse problem for the oscillatory Helmholtz equation was considered by A.N. Tikhonov [4].

In this work we present a formulation of the inverse problem that is different from those noted above. This formulation arises, for example, in the problems of interpreting geophysical data, and accounts with the possibility of utilizing the multifrequency of the given information.

Now we show on the example of one geophysical electrodynamic problem the way of reducing the latter considered inverse problem for the Helmholtz equation to the problem for the integral equation of the convolution type.

Let there be the homogeneous medium on the basis (with the constants ε_0, μ_0, σ_0 – dielectric constants, magnetic permeability, electric conductivity), which include a cylindrical body (with the other electromagnetic constants ε_1, μ_1, σ_1) with the generaterix parallel to axis OY. The **cross-section** Ω of this cylinder to the $\{y = 0\}$ coordinate plane **is unknown.**

Let E be polarized plane wave propagates from the upper half-space parallel to the $\{z = 0\}$ coordinate plane, dependence on the time is harmonic $\exp(i\omega t)$, the complex amplitude is $U_0(M) = \exp(ik_0 z)$, where $k_j = \frac{\mu\omega}{c}(\omega\varepsilon_j - i\sigma_j)$; $(j = 0, 1)$ – the wave number, c – the light velocity. The electrical and magnetic vectors of this plane wave are

$$\vec{E}_0(M) = U_0(M) \cdot \exp(i\omega t) \cdot \vec{Y}, \quad \vec{H}_0(M) = U_0(M) \cdot \exp(i\omega t) \cdot \vec{X}.$$

The electromagnetic field in this problem described by a pair of vectors $\vec{E}$, $\vec{H}$ satisfying Maxwell equations

$$\operatorname{rot} \vec{H} = \frac{1}{c}(i\varepsilon\omega\vec{E} + \sigma\vec{E}) = \frac{i}{c}\hat{\varepsilon}\omega\vec{E}, \quad \operatorname{rot} \vec{E} = \frac{1}{c} i\mu\omega\vec{H}$$

can be decomposed into fields of two types:

1. E-polarized: $\vec{E}(0, 0, E_z)$, $\vec{H}(H_x, H_y, 0)$

$$H_y = -\frac{ic}{\omega\mu} \cdot \frac{\partial E_z}{\partial x}, \qquad H_x = \frac{ic}{\omega\mu} \cdot \frac{\partial E_z}{\partial y},$$

2. H-polarized: $\vec{E}(E_x, E_y, 0)$, $\vec{H}(0, 0, H_z)$

$$E_y = \frac{ic}{\omega\hat{\varepsilon}} \cdot \frac{\partial H_z}{\partial x}, \qquad E_x = -\frac{ic}{\omega\hat{\varepsilon}} \cdot \frac{\partial H_z}{\partial y}.$$

The functions E_z end H_z are the solutions of the next boundary problems

$$\triangle_{xz} U(M) + k^2 U(M) = -U_0(M),$$
$$k = \begin{cases} k_0, & M \notin \Omega, \\ k_1, & M \in \Omega, \end{cases} \qquad U = (E_z, H_z).$$

The functions E_z, H_z, $i/\mu \cdot \partial E_z/\partial n$, $i/\hat{\varepsilon} \cdot \partial H_z/\partial n$ are continuity on the boundary surface of the inclusion $\partial\Omega$, and E_z, H_z are satisfied by the radiation conditions at infinity.

As it is shown by V.D. Kupradze, this differential problem is equivalent to the integral equation

$$U(M) = \frac{k_1^2 - k_0^2}{2\pi} \int_\Omega U(P)G(P;M)dv_P + U_0(M), \quad M \in \Omega, \tag{1}$$

where $G(P;M) = \pi/2i \cdot H_0^{(2)}[k_0 r(P;M)]$ – the Green function of the exterior space.

Moreover, this formula remains true for M lying in the exterior space.

If the norm of the integral operator in (1) is less than unit, it is possible to represent the solution of the equation (1) as the Neiman series which can be approached by it's first terms

$$U(M) \approx \frac{k_1^2 - k_0^2}{2\pi} \int_\Omega U_0(P)G(P;M)dv_P + U_0(M), \quad M \in \Omega. \tag{2}$$

For these conditions we have that the additional field $u(M) = U(M) - U(M_0)$ approximately satisfies the Helmholtz equation

$$\triangle u(M) + k_0^2 u(M) = -(k_1^2 - k_0^2)\exp(ik_0 z)\chi_\Omega(M), \tag{3}$$

where

$$\chi_\Omega(M) = \begin{cases} 1, & M \in \Omega, \\ 0, & M \notin \Omega \end{cases}$$

is the characteristic function of the set Ω.

Now we can formulate the next inverse problem.

On the basis of a family

$$u_{k_0}(x,0) = f_{k_0}(x), \qquad k \in A \tag{4}$$

of solutions to the direct problem for the equation (3), *known on the plane* $\{z = 0\}$ *(*A *is some set of wave numbers), it is required to determine the function* $\chi_\Omega(M)$.

In other formulation:

It is required to solve the parametric integral equation of the convolution type

$$L(k) \int_{R^2} \exp(ik_0\zeta)\chi_\Omega(\xi,\zeta)G(\xi,\zeta;x,0)d\xi d\zeta = f_k(x), \qquad k \in A,$$

with respect to the function $\chi_\Omega(M)$.

2. Uniqueness theorems for the integral equations

Let us consider the following problem:

Problem. *It is required to determine the function $u(Q)$ from the parametric equation*

$$\int_{R^n} u(Q)G(M_1 - Q, k)dv_Q = f(M_1, k), \quad k \in A, \quad M_1 \in R^{n-1}, \quad (5)$$

where $Q, M = (M_1, z) \in R^n$ are the points of n-dimensional Euclidean space; $z \in R$; k is the parameter ($k \in A \subset C$); dv_Q is an elementary volume of integrating near the point Q.

Theorem 1. *Let:*

1) *$f(M_1, k)$ be such, that there exists the function $u(Q)$ satisfying (5);*
2) *The kernel $G(Q, k)$ be such, that it is possible to apply the Fourier transform with respect to the variable $Q_1 \in R^{n-1}$, $Q = (Q_1, z)$, and its Fourier-image has the special form*

$$\int_{R^{n-1}} G(Q_1, z, k) \exp[-i(s, Q_1)]dv_{Q_1} = h(s, k) \exp[-zw(s, k)], \quad s \in R^{n-1}; \quad (6)$$

3) *A be a set of complex numbers having an accumulation point in a bounded portion of the complex plane;*
4) *$h(s, k) \neq 0$ for almost all s and $k \in A$;*
5) *$w(s, k)$ be an analytic and bounded in a neighborhood of the accumulation point of the set A, $\mathrm{Re}[w(s, k)] \geq \delta > 0$.*

Then a solution to the problem is unique on the class of complex-valued functions $u(Q)$, for which it is possible to apply the Fourier transform by Q_1 and the Laplace transform by z (for example, on the class of compactly supported functions from L_2).

Proof. Assume that $u(Q)$ cannot be uniquely determined from (5). Then there exit two different solutions of this equation. For their difference $\psi(Q) = u_1(Q) - u_2(Q) \not\equiv 0$ we obtain a family of homogeneous equations

$$\int_{R^n} \psi(Q)G(M_1 - Q, k)dv_Q = 0, \quad k \in A, \quad M_1 \in R^{n-1}.$$

Let us apply the Fourier transform with respect to the variable M_1, produce corresponding transformations by Fubini's theorem, and do the appropriate estimations. Then, after the substitution of the equation (6) we obtain

$$h(s,k)\int_0^\infty \tilde{\psi}(s,z)\exp[-z\,w(s,k)]dz = 0, \quad k \in A, \quad s \in R^{n-1}, \qquad (7)$$

where $\tilde{\psi}(s,k)$ is the Fourier transform of $\psi(Q)$ with respect to Q_1. In accordance with Condition 4 of the theorem we divide equation (7) by the function $h(s,k)$. Then, following Condition 5 we have that this integral is an analytic function by k allowing the analytic continuation from A. Hence, (7) is satisfied on the entire complex plane C. Then fixing s and making the substitution $p = w(s,k)$, we obtain

$$\int_0^\infty \tilde{\psi}(s,z)\exp(-zp)dz = 0, \quad \forall s, \quad \forall p \in C.$$

Using the uniqueness theorem for the Laplace transform we have $\tilde{\psi}(s,z) = 0$ for any s, and, applying the inverse Fourier transform on s to $\tilde{\psi}(s,z)$ we obtain $\psi(Q) = 0$. This contradiction proves the theorem. □

When the set A does not have the finite accumulation point, we can obtain the uniqueness theorems for some partial cases of functions $h(s,k)$ and $w(s,k)$, which appeared in applied problems.

Theorem 2. *Let:*

1) *Conditions* 1, 2, 4 *of Theorem* 1 *fulfill;*
2) *A solution of the equation* (5) *belongs to the class of compactly supported complex-valued function from* L_2, *and*

$$\operatorname{supp} U(Q) = \Omega \subset \{Q = (Q_1, z) \in R^m : \; 0 < \delta \le z \le a\}, \quad m = 2, 3;$$

3) $A = \{k_n : \; |k_n| = \frac{n\pi}{a}, \; n \in \mathbb{N}\} \subset R$;
4) $w(s,k) = (s^2 - k^2)^{1/2}$;

Then a solution to the problem is unique.

Theorem 3. *Let:*

1) *Conditions* 1, 2 *of Theorem* 2 *fulfill;*
2) $A = \{k_n : \; |k_n| = n\pi/2a, \; n \in \mathbb{N}\} \subset R$;
3) $w(s,k) = ik + (s^2 - k^2)^{1/2}$.

Then a solution to the problem is unique.

Theorem 4. *Let:*

1) *Conditions* 1,2 *of Theorem* 2 *fulfill, but* $u(Q)$ *is the real function and* $\Omega \subset \{0 < \delta \leq z\} \subset R^2$*;*
2) A *be a set of complex numbers* k *such that the set of points* $\{ik, -ik\}_{k\in A}$ *contains a countable subset of numbers* $\{\lambda_n\}_1^\infty$, $\mathrm{Re}\,\lambda_n > 0$, *such that the series*

$$\sum_{n=1}^{\infty} \left(1 - |\lambda_n|^2\right)^{-1} \mathrm{Re}\,\lambda_n$$

diverges.

Then a solution to the problem is unique.

For the illustration of the method, with the help of which we obtain Theorems 2–4, we give the proof of Theorem 3 in 3-dimensional case.

Proof. All considerations which bring about the family of equations (7) in the proof of Theorem 1 are the same. Furthermore, we write down the function $\exp[-zw(s,k)]$ as

$$\exp[-zw(s,k)] = \exp(-izk)\exp[-z(s_1^2 + s_2^2 - k^2)^{\frac{1}{2}}]$$

and decompose $\exp[-z(s_1^2 + s_2^2 - k^2)^{\frac{1}{2}}]$, as the function of argument $z(s_1^2 + s_2^2 - k^2)^{\frac{1}{2}}$, in the series

$$\exp\left[- z(s_1^2 + s_2^2 - k^2)^{\frac{1}{2}}\right] = -(s_1^2 + s_2^2 - k^2)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(s_1^2 + s_2^2 - k^2)^n z^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(s_1^2 + s_2^2 - k^2)^n z^{2n}}{(2n)!} \tag{8}$$

converging on all complex plane. There s_1 and s_2 are the Fourier transform parameters which correspond to the variables x and y on the plane $\{z = 0\}$. Since $\tilde{\psi}(s_1, s_2, z)$ is the Fourier transform of the compactly supported function, it is an entire function (it follows from Paly-Winer's theorem). So we can write down the expression $\tilde{\psi}(s_1, s_2, z)\exp(-izk)$ as the series

$$\tilde{\psi}(s_1, s_2, z)\exp(izk) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} s_1^{l-m} s_2^m \left[\phi_{lm}(z,k) + i\hat{\phi}_l m(z,k)\right], \tag{9}$$

which converges for all s_1, s_2 (uniformly for all $z \in [0,a]$). Here

$$\phi_{lm}(z,k) = \begin{cases} \psi_{lm}(z)\cos(kz), & l \text{ - even}, \\ \psi_l m(z)\sin(kz), & l \text{ - odd}. \end{cases}$$

$$\hat{\phi}_{lm}(z,k) = \begin{cases} -\psi_{lm}(z)\sin(kz), & l \text{ - even}, \\ \psi_l m(z)\cos(kz), & l \text{ - odd}. \end{cases}$$

$$\psi_{lm}(z) = \frac{1}{m!(l-m)!}\int_{R^2}\psi(x,y,-z)x^{l-m}y^m\,dxdy \cdot \begin{cases} (-1)^{l/2}, & l \text{ - even}, \\ (-1)^{l+1/2}, & l \text{ - odd}. \end{cases}$$

It is not difficult to show that the functions $\psi_{lm}(z)$ $(l,m = 0,1,\ldots,\infty)$ are the elements of the class $L_2[0,a]$. Since series in (8) converge with respect to z on all complex plane, then changing the order of integration and summation we have

$$-\sum_{n=0}^{\infty}(s_1^2+s_2^2-k^2)^n\int_0^a \tilde{\psi}(s_1,s_2,z)\frac{z^{2n+1}}{(n+1)!}\,dz +$$
$$(s_1^2+s_2^2-k^2)^{-\frac{1}{2}}\sum_{n=0}^{\infty}(s_1^2+s_2^2-k^2)^n\int_0^a \tilde{\psi}(s_1,s_2,z)\frac{z^{2n}}{(2n)!}dz = 0, \quad k \in A.$$

Denoting $G_1(s_1,s_2,k)$ and $G_2(s_1,s_2,k)$ as corresponding series here, we obtain

$$G_1(s_1,s_2,k) + (s_1^2+s_2^2-k^2)^{-\frac{1}{2}}G_2(s_1,s_2,k) = 0, \quad k \in A. \tag{10}$$

From uniform converges with respect to $z \in [0,a]$ of the series (9) for s_1, s_2, we have that $G_1(s_1,s_2,k)$ and $G_2(s_1,s_2,k)$ are the entire functions for s_1, s_2, and, therefore, they have not branch points for all finite s_1, s_2. Then from (10) it follows two systems of equations:

$$G_1(s_1,s_2,k) = \sum_{n=0}^{\infty}(s_1^2+s_2^2-k^2)^n\int_0^a \tilde{\psi}(s_1,s_2,z)\frac{z^{2n+1}}{(2n+1)!}dz = 0, \quad k \in A,$$
$$G_2(s_1,s_2,k) = \sum_{n=0}^{\infty}(s_1^2+s_2^2-k^2)^n\int_0^a \tilde{\psi}(s_1,s_2,z)\frac{z^{2n}}{(2n)!}dz = 0, \quad k \in A.$$

If it is not valid, then entire function $G_1(s_1,s_2,k)$ must have the branch point for $s_1^2+s_2^2 = k^2$, but it is impossible.

Using the expression (9) in (10) and corresponding transformations of power series, we obtain the system of equations:

$$\sum_{q\geq 0, n-2q\geq 0}\ \sum_{r\geq 0, v-2r\geq 0} \int_0^a \phi_{u+v-2(q+r),u-2q}(z,k) g_{jr,q}(k,z)dz = 0,$$

$$\sum_{q\geq 0, n-2q\geq 0}\ \sum_{r\geq 0, v-2r\geq 0} \int_0^a \hat{\phi}_{u+v-2(q+r),u-2q}(z,k) g_{jr,q}(k,z)dz = 0, \quad (11)$$

$$k \in A; \quad u, v = 0, 1, \ldots, \infty; \quad j = 1, 2,$$

where

$$g_{1\,r,q}(k,z) = \frac{z^{2(r+q)}}{k\, r!q!} \sum_{n=0}^{\infty} (-1)^n \frac{(n+r+q)!}{n![2(n+r+q)+1]!} (kz)^{2n+1}$$

$$g_{2\,r,q}(k,z) = \frac{z^{2(r+q)}}{r!q!} \sum_{n=0}^{\infty} (-1)^n \frac{(n+r+q)!}{n![2(n+r+q)]!} (kz)^{2n}$$

$$r, q = 0, 1, \ldots, \infty,$$

and it is important that

$$g_{1\,0,0}(k,z) = k^{-1}\sin(kz), \quad g_{2\,0,0}(k,z) = \cos(kz).$$

From (11) we have for $u = v = 0$

$$\int_0^a \psi_{00}(z)\cos(kz)\sin(kz)dz = 0, \quad k \in A,$$

$$\int_0^a \psi_{00}(z)\sin^2(kz)dz = 0, \quad k \in A,$$

$$\int_0^a \psi_{00}(z)\cos^2(kz)dz = 0, \quad k \in A,$$

$$\int_0^a \psi_{00}(z)\sin(kz)\cos(kz)dz = 0, \quad k \in A.$$

From these equations follows

$$\int_0^a \psi_{00}(z)dz = 0, \quad \int_0^a \psi_{00}(z)\sin(2kz)dz = 0 \quad k \in A.$$

As the system of functions $\{\sin(nx)\}_{n\in N}$ is total in $L_2[0,\pi]$, then $\psi_{00}(z) = 0$. Since from (11) requrently follows the analogous expressions for all functions $\psi_{l\,m}(z)$, $(l, m = 0, 1, \ldots, \infty)$, then we have that $\psi_{l\,m}(z) = 0$, $(l, m = 0, 1, \ldots, \infty)$. Therefore, from (9) we obtain $\tilde{\psi}(s_1, s_2, z) = 0$ for all s_1, s_2. After applying the inverse Fourier transform to $\tilde{\psi}(s_1, s_2, z)$ with respect to s_1, s_2 we have $\psi(x,y,z) = 0$. This contradiction proves the theorem. □

In conclusion let us note that this article continues the author's works [5–8] in which the correct formulation of the inverse problems of the theory of wave propagation in the complicated medium on the basis was studied.

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