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Modeling of barotropic fluid dynamics on the sphere based on the contour dynamics

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In this paper, a solution of equations of the barotropic fluid evolution in the spherical geometry by the method of the contour dynamics is considered. Integro-differential equations for velocity and some dynamic characteristics are obtained. The motions and interactions of some vortex structures on the sphere are studied.

1. Introduction

The contour dynamics (CD) as a technique of solving nonlinear two-dimensional Euler equations in classical hydrodynamics was first described by Zabusky, Hughes and Roberts in 1979. This method is applied for modeling of the evolution of vortical structures and is based on an assumption that the vorticity has piecewise-constant distribution. Its main advantage is that the problem is reduced to the dynamics of contours. So, the dimension of the field of states of the system decreases. In present, there are many publications which are concerned with the method of contour dynamics in various fields – from the physics of plasma to dynamics of the atmosphere and ocean. One can find a detailed review of studies which presents the contour dynamics in hydrodynamics problems in [1].

The present study describes a numerical scheme, which permits us to apply the CD to the barotropic fluid evolution problem on the sphere.

2. The contour dynamics for the atmosphere and ocean on the sphere

2.1. General principles of the CD. Sufficient conditions of using the CD can be formulated as the following five principles:

1. The fluid motion is quasi-two-dimensional, and the velocity field can be written as

$$u = u_0(x,y,t) - a(x,y,t)\psi_y, \quad v = v_0(x,y,t) + a(x,y,t)\psi_x. \qquad (1)$$

The velocity is written as a total of a "background" value (u_0, v_0) and a component connected with a stream function ψ. The coefficient a depends on the geometry of a problem.

2. As the corollary of the dynamic equation, there exists some adiabatic invariant $\Pi(x, y, t)$ which is constant for any material point:

$$\frac{d\Pi}{dt} \equiv \Pi_t + u\Pi_x + v\Pi_y = 0. \tag{2}$$

3. The adiabatic invariant is connected with the stream function by an elliptic operator L:

$$\Pi = L\psi. \tag{3}$$

4. The operator L is reversed by the Green function $G(x, y, \xi, \eta)$:

$$\psi = \iint \Pi(\xi, \eta, t) G \, d\xi \, d\eta. \tag{4}$$

5. At the initial moment

$$\Pi = \Pi_0 \chi(D_0), \tag{5}$$

where Π_0 is a constant. When moving, D_0 transfers into D with a boundary $C(t)$. The validity of the expression $\Pi = \Pi_0\chi(D)$ at any moment of time is evident from the condition of the invariant conservation.

We determine the function

$$F(x, y, \xi, \eta) = \int_0^1 G(x, y; x + (\xi - x)z, y + (\eta - y)z) z \, dz, \tag{6}$$

which satisfies the equality $G = [(\xi - x)F]_\xi + [(\eta - y)F]_\eta$.

Now, taking this property into account and applying the Stokes theorem, we obtain the formula for the stream function

$$\psi(x, y, t) = \Pi_0 \oint_{C(t)} F[(\xi - x)d\eta - (\eta - y)d\xi]. \tag{7}$$

Thus, the stream function and the velocity field (see (1)) are unambiguously determined by the contour $C(t)$, whose motion is described by the equations

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \tag{8}$$

where x, y are meant to denote the Lagrangian co-ordinates of the fluid particles belonging to the contour.

In case of an arbitrary piecewise-constant distribution, the velocity field is calculated by superposition of effects of all the contours in which adiabatic invariant has a discontinuity.

A simple case, where conditions of applicability of the method are fulfilled is a fluid motion described by the Euler equations.

2.2. Numerical solution of the Euler equations. Under the assumption that fluid is incompressible, inviscid and its motion is plane-parallel, two problems are considered: interaction of an initially round vortex of radius 0.3 and a point vortex with the same sign and with equal values of the circulation -0.09π and interaction of two round vortices of radius 0.3 and with the circulation -0.09π.

The evolution of vortices is described by the Euler system of equations which can be written down in terms of the stream function and the vorticity.

$$\omega_t + u\omega_x + v\omega_y = 0, \quad \psi_{xx} + \psi_{yy} = \omega, \quad u = -\psi_y, \quad v = \psi_x.$$

The expression for the stream function is obtained using the Poisson equation

$$\psi = \frac{1}{2\pi} \iint \omega(\xi, \eta) \ln R \, d\xi \, d\eta,$$

and, respectively,

$$\vec{U} = \frac{\omega}{2\pi} \oint_C \ln R(e_x \, d\xi + e_y \, d\eta). \tag{9}$$

When solving problems, we made use of the numerical scheme proposed in [2].

In Figure 1, there is shown an interaction between the round vortex and the point vortex. The vortical domain and the point vortex spin around the point in the centre of the flow. When moving, the shape of the domain becomes oval. At $t = 24$, a quasireturn occurs – the shape of the domain becomes nearly round.

Two round domains rotate around the centre of symmetry, in the time interval from 4 to 12 they are stretching, becoming closer to each other, at $t = 15$ their boundaries practically coincide. The motion of the domains is displayed in Figure 2.

It is impossible to solve the problem of interaction of two vorticity domains at $t > 15$, because the shapes of the contours are becoming more complicated and the approximation is not accurate enough.

In the case of interaction of the vortical domain and the point vortex, a relative alteration of the circulation $\Delta\Gamma/\Gamma$ was calculated. In the table, Δt is the number of nodes, Δt is a time step, all the values are taken at $t = 12$.

N	20	50	20	50	250
Δt	1	1	0.1	0.1	1
$\Delta\Gamma/\Gamma$	$1.46 \cdot 10^{-1}$	$1.51 \cdot 10^{-1}$	$1.08 \cdot 10^{-4}$	$4.30 \cdot 10^{-5}$	$1.51 \cdot 10^{-1}$

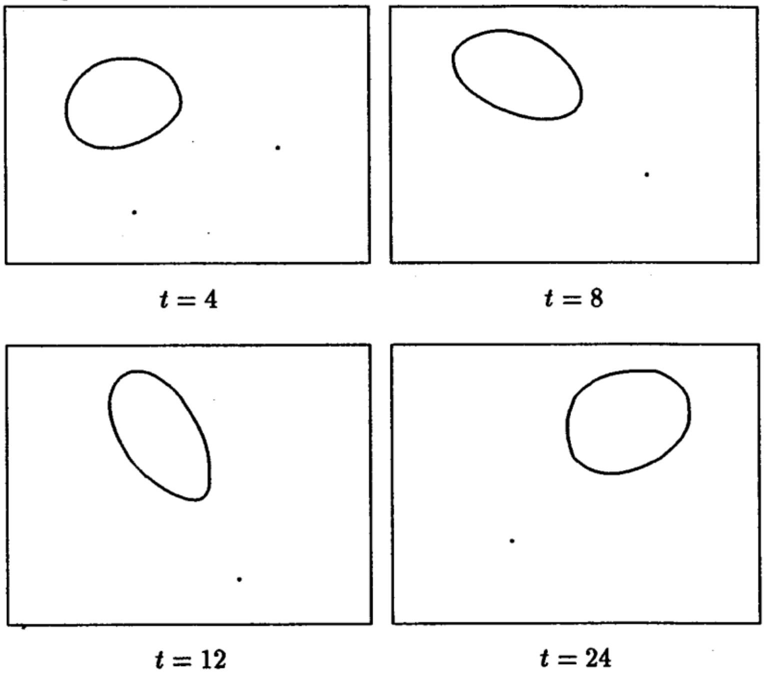

Figure 1. Plain geometry. The interaction between a vortical region and a point vortex

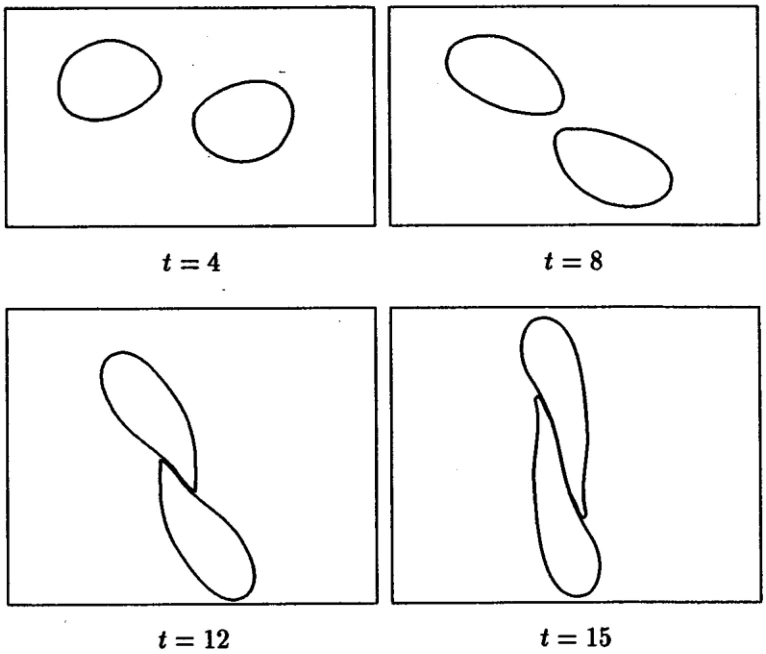

Figure 2. Plain geometry. The interaction between two vortical regions

So, we can see that although the scheme is not dissipative, it can be applied to the investigation of motions of the Euler fluid at a certain time interval.

2.3. The formulation of the problem in the case of spherical geometry. In order to observe the flow on the sphere, we introduce a modified spherical co-ordinate system (r, λ, μ) with the unit vectors i_r, i_λ, i_μ:

$$r = \sqrt{x^2 + y^2 + z^2} \text{ (radius)}, \quad \lambda = \text{arctg}\,\frac{y}{x} \text{ (longitude)},$$

$$\mu = \sin\varphi = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \text{ (φ is latitude)}.$$

Let us consider a barotropic model, in which the vertical component is small as compared to the meridional and the zonal components (u and v, respectively). The vector field of the curl looks like $\omega = \omega(\lambda, \mu, t)i_r$, we neglect the zonal component of the curl. Under these conditions, the equations of the evolution are written as

$$\frac{d\lambda}{dt} = \frac{v}{\sqrt{1-\mu^2}}, \quad \frac{d\mu}{dt} = u\sqrt{1-\mu^2}. \tag{10}$$

The equation of continuity is expressed in terms of the variables (λ, μ):

$$\frac{\partial}{\partial\lambda}\left(\frac{u}{\sqrt{1-\mu^2}}\right) + \frac{\partial}{\partial\mu}\left(\sqrt{1-\mu^2 v}\right) = 0.$$

This relation allows us to introduce a general stream function Ψ, which satisfies the conditions

$$\frac{\partial\Psi}{\partial\lambda} = \sqrt{1-\mu^2 v}, \quad \frac{\partial\Psi}{\partial\mu} = -\frac{1}{\sqrt{1-\mu^2}}u. \tag{11}$$

In the model, the vorticity conservation law is fulfilled:

$$\frac{d\omega}{dt} = 0, \tag{12}$$

and the stream function and the vortex are connected to each other by the Laplace operator $\omega = \Delta\Psi$.

Solving the Poisson equation with respect to Ψ, we obtain the formula:

$$\Psi(\vec{x}) = \frac{1}{2\pi}\iint\limits_S \omega \ln(r(\vec{x}, \vec{y})/2)\, ds_1, \tag{13}$$

where r is the distance between the point of integration $\vec{y}$ and the point $\vec{x}$, in which we calculate the value of the stream function.

The solution to the Poisson equation exists if the following condition limiting the class of functions ω is fulfilled

$$\int_S \omega\, ds = 0. \tag{14}$$

Let $\{D_i(t)\}$, $i = 1, \dots, n$, be a set of domains, to any of the domains there corresponds a value of the vortex ω_i and outside these domains the vortex is equal to zero. Then we have the expressions for the velocity:

$$u = -\frac{\sqrt{1-\mu^2}}{2\pi R} \sum_{i=1}^{n} \iint_{D_i} \omega_i \frac{\partial \ln(r(\vec{x}, \vec{y})/2)}{\partial \mu}\, ds,$$

$$v = -\frac{1}{2\pi R\sqrt{1-\mu^2}} \sum_{i=1}^{n} \iint_{D_i} \omega_i \frac{\partial \ln(r(\vec{x}, \vec{y})/2)}{\partial \lambda}\, ds.$$

Taking into account the fact that $\partial \ln r/\partial\lambda = -\partial \ln r/\partial\lambda'$ and applying the Stokes theorem, we obtain the equation

$$-\frac{1}{2\pi R\sqrt{1-\mu^2}} \int_{D_i} \omega_i \frac{\partial \ln(r(\vec{x}, \vec{y})/2)}{\partial \lambda'} ds = -\frac{1}{2\pi R\sqrt{1-\mu^2}} \int_{C_i} \omega_i \ln\frac{r(\vec{x}, \vec{y})}{2} d\mu',$$

where C_i is the boundary of the domain D_i.

We take the integral over D_i in the expression for the meridional velocity using the formula

$$-\frac{\sqrt{1-\mu^2}}{2\pi R} \iint_{D_i} \omega_i \frac{\partial \ln(r(\vec{x}, \vec{y})/2)}{\partial \mu} ds = -\frac{\sqrt{1-\mu^2}}{2\pi R} \int_{C_i} F\omega_i d\lambda',$$

where $F = \int \frac{\partial \ln(r(\vec{x},\vec{y})/2)}{\partial\mu} \partial\mu'$. The function F is determined accurate within an additive function depending on the parameters μ, λ, λ' whose integral along the contour C is equal to zero.

Therefore, we obtain the final expressions for the velocity:

$$u = -\frac{\sqrt{1-\mu^2}}{2\pi R} \sum_{i=1}^{n} \int_{C_i} F\omega_i\, d\lambda', \tag{15}$$

$$v = -\frac{1}{2R\pi\sqrt{1-\mu^2}} \sum_{i=1}^{n} \int_{C_i} \omega_i \ln\frac{r(\vec{x}, \vec{y}}{2} d\mu'. \tag{16}$$

The projections of the velocity are displayed as contour integrals of certain functions. Formulas (15), (16) allow us to use a numerical algorithm which is similar to the contour dynamics for plane motions of the ideal inviscid fluid to solve problems of the class under consideration.

3. Numerical experiment

Problems of the fluid motion with different initial distributions of the vorticity were examined. In all the cases, the vorticity field is antisymmetric with regard the equator. It permits the accomplishment of condition (14). The vorticity field remains antisymmetric for all the time of evolution, therefore it is sufficient to compute the velocity for the nodes located in the north hemisphere.

The evolution of the contours is described by Lagrangian equations (10). The contours are approximated by several nodes connected by the curves $\lambda = \lambda_1 + (\lambda_2 - \lambda_1)s$, $\mu = \mu_1 + (\mu_2 - \mu_1)s$, where (λ_1, μ_1) and (λ_2, μ_2) are the co-ordinates of the nodes and the parameter s runs through the values from zero to one.

In case when integration is conducted along the segment of the curve neighboring to the point $\vec{x}$, the function $\ln(r)$ is approximated by the function $((1-\mu^2)(\lambda - \lambda')^2 + \frac{1}{1-\mu^2}(\mu - \mu')^2)/2$, a relative error of the integral is of the first order with respect to distance between the neighboring nodes Δx. The function F from equation (15) has also the logarithmic singularity at $\vec{x} = \vec{y}$, therefore the value of the integral along the interval neighboring to the point $\vec{x}$ is taken with respect to the value in the middle point.

In other cases, the integrals are computed by the method of the trapezoids. The function F is determined by the condition $F(\lambda, \mu, \lambda', \mu') = 0$ if $\lambda \neq \lambda'$ and the point of integration (λ', μ') lies on the north hemisphere, and it is determined by the condition $F(\lambda, \mu, \lambda', -\mu) = 0$ if $\lambda \neq \lambda'$ and (λ', μ') is the point lying on the south hemisphere. Specified like that the initial level of F allows us to approximately calculate the integral with low computer cost. The interval with the end points (λ', μ') and $(\lambda', \pm\mu)$ (the sign "+" is taken for the point located on the north hemisphere and " - " is taken if $(\lambda', \mu'$ is the point symmetric to one of the nodes and lying on the south hemisphere) is divided into the parts whose lengths do not exceed a given value. The integral is calculated with the use of values in the middle points of these intervals. The equation of the evolution is approximated by the scheme of the predictor–corrector type:

$$\lambda^{j+0.5} = \lambda^j + \frac{u(\lambda^j, \mu^j)}{\sqrt{1-(\mu^j)^2}} \frac{\Delta t}{2}, \qquad \lambda^{j+1} = \lambda^j + \frac{u(\lambda^{j+0.5}, \mu^{j+0.5})}{\sqrt{1-(\mu^{j+0.5})^2}} \Delta t. \quad (17)$$

This scheme is of the second approximation order with respect to time. It is stable.

Solving the problem of interaction of two domains, where the vorticity is equal to 0.1, we obtain the picture of the motion analogous to the dynamics of two vortices in the classical Euler fluid (Figure 3): the domains are rotating around the point located at the centre of the flow. In the time interval

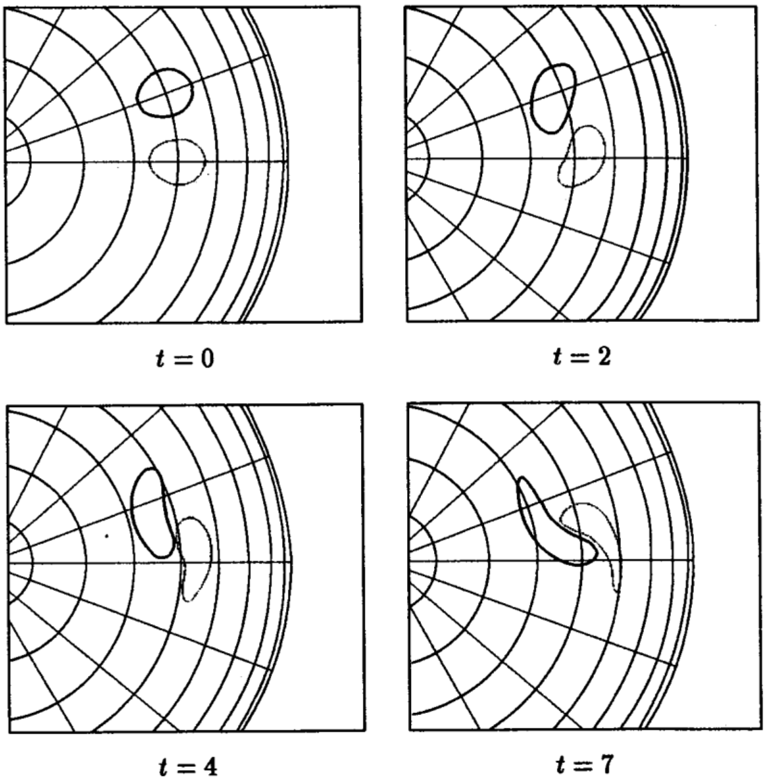

Figure 3. The interaction between two regions with equal vorticity

from 0 to 40 their shape is becoming oval. At $t = 40$, the domains are stretched along the parallel. At $t = 70$, the segments of vortices boundaries practically coincide. In the following, one can consider the system as vortex of the round shape with two bands of vorticity, emanating from it.

The interaction of a vortical domain and a point vortex is displayed in Figure 4. The stream function for the motion induced by the point vortex is obtained by approaching to a limit in the expression of the stream function of the velocity field induced by a vorticity domain: $\Psi = \frac{\Gamma}{2\pi}\ln(r)$, where Γ is circulation of the point vortex and r is a distance between the vortex and the evaluation point.

The boundary of the vortical domain is given by the equations $\lambda = \frac{\pi}{27}\cos(x)$, $\mu = \sin(\frac{5\pi}{18} + \frac{\pi}{27}\sin(x))$, $x \in [0.2\pi]$, the initial co-ordinates of the point vortex $\lambda = \frac{\pi}{18}$, $\mu = \frac{5\pi}{18}$. When the circulation of a point vortex takes value 0.01 and vorticity is equal to unit at any point of the domain, the point vortex rotates around vorticity domain, the shape of the domain changes insignificantly. If circulation is increased to 0.05, the shape of the domain will change to a greater extent and the centre of rotation will move towards the point vortex. The domain stretches along the parallel when the

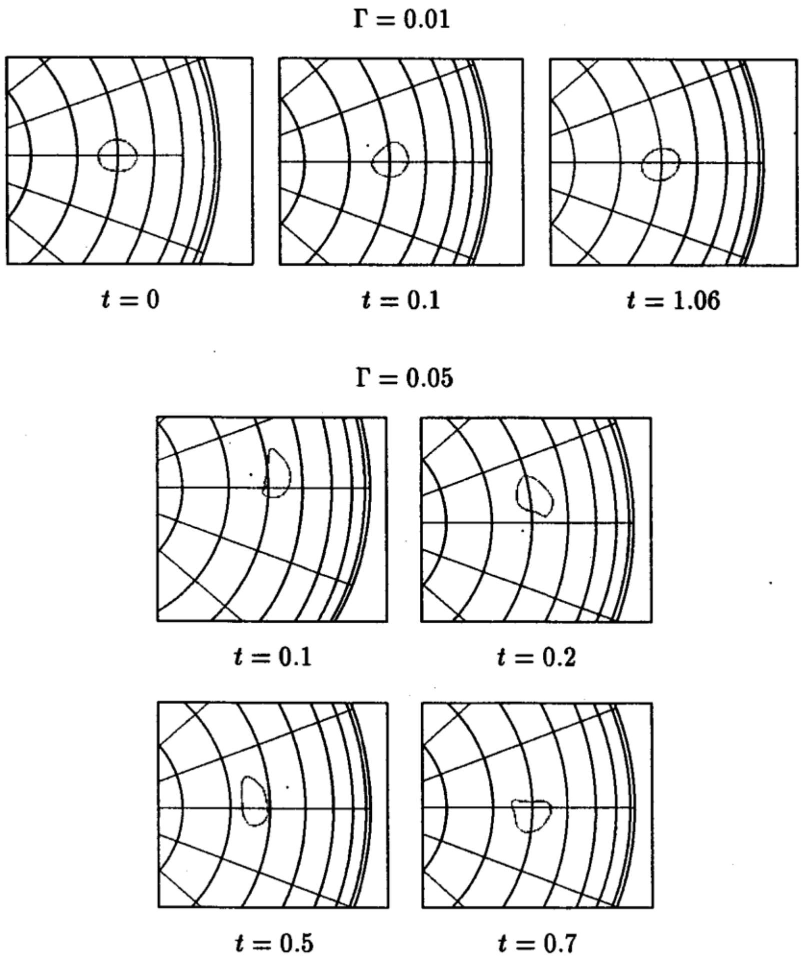

Figure 4. The interaction between a vortical region and a point vortex

vortex is located toward the north from the domain. When the system has performed a turn of 180°, the shape of the domain become almost triangular. The period of rotation around the centre of the system decreases to 0.7. It is obvious that the distance between the point vortex and the domain has increased. It indicates to the fact that the scheme is non-conservative.

Also, we have considered the problems with allowance for rotation of the sphere. Velocity of the point rotating with the sphere is determined by $u = \sqrt{1-\mu^2} i_r$. The planetary vorticity is equal to 2μ, respectively, and the adiabatic invariant is the absolute curl equal to the sum of relative and planetary curls. In this case, the motion of vortex is obtained by the superposition of the solution of a similar problem for a relative curl and the shift in the direction opposite to rotation, the speed of the shift being equal to the speed of rotation.

4. Conclusion

In general, we can say that the results of the numerical experiment do not contradict to our empiric notions. So the contour dynamics is applicable to studying local vortical structures on the sphere. Its main benefit consists in fact that a two-dimensional problem is reduced to a one-dimensional problem and integrals taken over regions are reduced to integrals along their boundaries. Another advantage of this method is that it allows us to solve the problems with non-smooth initial data distributions.

Often in the process of the evolution, contours elongate and complicate their shape. In this case, these CD algorithms become not applicable. To avoid such difficulties, there exist modifications of the method which contain nodes redistribution on the contour and cut-off of dynamically insignificant segments (so-called "the contour surgery", [3]).

The applied algorithm allows us to describe the motion of vortices only qualitatively. The main difficulties are connected with using unbounded integrands. The approximation can be violated, in particular, in the case when the region has a non-smooth boundary. To improve the approximation we can invent a procedure of nodes redistribution. The applied difference scheme is not conservative, but one can achieve the conservation of invariants with necessary accuracy, by decreasing of a time step. Further it is possible to construct modifications of the method similar to "the contour surgery" for plane flows.

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